Problems 1.2 Conditional and Biconditional Connectives

Working Definitions: The following definitions are needed in some problems in this and if following sections.

- An integer n **divides** an integer m (and we write $n|m$) if there exists an integer q such that $m = n \times q$.
- An integer n is even if there exists an integer k such that $n = 2k$.
- An integer n is odd if there exists an integer k such that $n = 2k + 1$.
- A natural number p is prime if it is only divisible by 1 and itself.

1. Identify the assumption and conclusion in the following conditional sentences and tell if the implication is true or false.

a) If pigs fly then I am richer than Bill Gates.
Ans: Assumption: pigs fly, Conclusion: I am richer than Bill Gates, True

b) If a person got the plague in the 17^{th} century they were in trouble.
Ans: Assumption: a person got the plague in the 17^{th} century
Conclusion: they were in trouble, True

c) If you miss class over 75% of the time you are in trouble.
Ans: Assumption: you miss class over 75% of the time, Conclusion: you are in trouble, True

d) If x is a prime number then x^2 is prime too.
Ans: Assumption: x is a prime number Conclusion: x^2 is prime too, False

e) If x and y are prime numbers, then so is $x + y$.
Ans: Assumption: x and y are prime numbers Conclusion: $x + y$ is prime, False

f) If the determinant of a matrix is nonzero, the matrix has an inverse.
Ans: Assumption: If the determinant of a matrix is nonzero, Conclusion: the matrix has an inverse, True

g) If f is a 1-1 function then f has an inverse.
Ans: Assumption: f is a 1-1 function, Conclusion: f has an inverse, True

2. Write the contrapositive of the conditional sentences in Problem 1.
Ans:

a) If I am not richer than Bill Gates, then pigs do not fly.

b) If people in the 17th century were not in trouble, then they never got the plague.

c) If you are not in trouble, then you did not miss class 75% of the time.
d) If x^2 is not prime, then x is not prime.
e) If $x+y$ is not prime, then either x is not prime or y is not prime.
f) If a matrix does not have an inverse, then the determinant of the matrix is zero.
g) If f does not have an inverse, then f is not a 1-1 function.

3. Let P be the sentence "$4>6$", Q the sentence "$1+1=2$", and R the sentence "$1+1=3$". What is the truth value of the following sentences?

a) $P\wedge\sim Q$	**Ans:** F
b) $\sim(P\wedge Q)$	**Ans: T**
c) $\sim(P\vee Q)$	**Ans: F**
d) $\sim P\wedge\sim Q$	**Ans: T**
e) $P\wedge Q$	**Ans: F**
f) $P\Rightarrow Q$	**Ans: T**
g) $Q\Leftrightarrow R$	**Ans: F**
h) $P\Rightarrow(Q\Rightarrow R)$	**Ans: T**
i) $(P\Rightarrow Q)\Rightarrow R$	**Ans: F**
j) $(R\vee Q\vee R)\Leftrightarrow(P\wedge Q\wedge R)$	**Ans: F**

4. Let P be the sentence “Jerry is richer than Mary”, Q is the sentence “Jerry is taller than Mary”, and R is the sentence “Mary is taller than Jerry.” For the following sentences what can you conclude about Jerry and Mary if the sentences are true. Express the information in a convenient form.

a) $P\vee Q$

Ans: Either P is true or Q is true (or both are true).

b) $P\wedge Q$

Ans: Both P and Q are true.

c) $\sim P\vee Q$

Ans: Either P is false or Q is true

d) $Q\wedge R$

Ans: Both Q and R are true.

e) $\sim Q \wedge \sim R$

Ans: Both Q and R are false.

f) $P \wedge (P \Rightarrow Q)$

Ans: Both P and Q are true.

g) $P \Leftrightarrow (Q \vee R)$

Ans: Either all P,Q,R are true or P is false and one (or both) Q and R are false.

h) $Q \wedge (P \Rightarrow R)$

Ans: Either all P,Q,R are true or Q is false and P is true and R false.

i) $P \vee Q \vee R$

Ans: Either one of P,Q,R are true.

j) $P \vee (Q \wedge R)$

Ans: Either P is true or both Q and R are true.

5. Construct truth tables to show the following sentences mean the same thing.

a) P iff Q means the same as $\sim P$ iff $\sim Q$

Ans:

P	Q	$P \Leftrightarrow Q$	$\sim P$	$\sim Q$	$\sim P \Leftrightarrow \sim Q$
T	T	**T**	F	F	**T**
T	F	**F**	F	T	**F**
F	T	**F**	T	F	**F**
F	F	**T**	T	T	**T**

b) $\sim (P \Leftrightarrow Q)$ means the same as $(P \wedge \sim Q) \vee (\sim P \wedge Q)$

Ans:

P	Q	$P \Leftrightarrow Q$	$\sim (P \Leftrightarrow Q)$	$\sim P$	$\sim Q$	$P \wedge \sim Q$	$\sim P \wedge Q$	$(P \wedge \sim Q) \vee (\sim P \wedge Q)$
T	T	T	**F**	F	F	F	F	**F**
T	F	F	**T**	F	T	T	F	**T**
F	T	F	**T**	T	F	F	T	**T**
F	F	T	**F**	T	T	F	F	**F**

c) $P \Rightarrow Q$ means the same as $\sim P \vee Q$

Ans:

P	Q	$P \Rightarrow Q$	$\sim P$	$\sim P \vee Q$
T	T	**T**	F	**T**
T	F	**F**	F	**F**
F	T	**T**	T	**T**
F	F	**T**	T	**T**

6. Translate the given sentences in English to conditional form.

a) Unless you study you won't get a good grade.
Ans: If you do *not* study, then you will *not* get a good grade, or its contrapositive, which states: if you *did* get a good grade then you studied.

b) Do you like it? It's yours."
Ans: If you like it, then it is yours.

c) Come here and I'll help you.
Ans: If you come here, then I'll help you.

d) Get out or I'll call the cops.
Ans: If you do not get out, then I will call the cops. (Normally one drops the "then" in normal conversation.)

e) Anyone who doesn't study deserves to flunk.
Ans: If anyone does not study, then they deserve to flunk.

f) Criticize her and she will slap you.
Ans: If you criticize her, (then) she will slap you. (Often people drop the "then.")

g) With his toupee on the professor looks younger.
Ans: If the professor wears his toupee, he looks better.

7. **(In Plain English)** Without making a truth table, say why the following are true.

a) $\left[(P \vee Q) \wedge \sim P\right] \Rightarrow Q$

Ans: If we know that P or Q but that P is not true, then we have no choice but to assume Q is true.

b) $\left[P \wedge (Q \wedge \sim Q)\right] \Rightarrow \sim P$

Ans: If by assuming P and from this you deduce a contradiction, then you must assume your assumption P is false.

c) $(P \vee Q) \Rightarrow (\sim P \Rightarrow Q)$

Ans: $P \vee Q$ says that P or Q is true. Hence, if P is not true then Q are true.

8. **(Distributive Laws for AND and OR)** For the sentences P, Q and R verify the **distributive laws**

a) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

Ans: Note that the columns (2) and (5) are the same.

			(1)	(2)	(3)	(4)	(5)
P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**T**	T	F	**T**
T	F	T	T	**T**	F	T	**T**
T	F	F	F	**F**	F	F	**F**
F	T	T	T	**F**	F	F	**F**
F	T	F	T	**F**	F	F	**F**
F	F	T	T	**F**	F	F	**F**
F	F	F	F	**F**	F	F	**F**

b) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Ans: Note that the columns (2) and (5) are the same.

			(1)	(2)	(3)	(4)	(5)
P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	**T**	T	T	**T**
T	T	F	F	**T**	T	T	**T**
T	F	T	F	**T**	T	T	**T**
T	F	F	F	**T**	T	T	**T**
F	T	T	T	**T**	T	T	**T**
F	T	F	F	**F**	T	F	**F**
F	F	T	F	**F**	F	T	**F**
F	F	F	F	**F**	F	F	**F**

9 **(Inverse, Converse, and Contrapositive)** One of the following sentences has the same meaning as $P \Rightarrow Q$. Which one is it?

$$\textit{inverse}: \sim P \Rightarrow \sim Q$$

$$\textit{converse}: Q \Rightarrow P$$

$$\textit{contrapositive}: \sim Q \Rightarrow \sim P$$

For the two sentences which are not always true, give examples where they are true.

10. **(True or False?)** Is the following statement ever true? Is it ever false?

$$\left[(P \Rightarrow Q) \wedge Q\right] \Rightarrow P$$

Ans: From the truth table

		(1)	(2)	(3)
P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge Q$	$\left[(P \Rightarrow Q) \wedge Q\right] \Rightarrow P$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	T	**F**
F	F	T	F	**F**

the statement is TRUE when P is true, otherwise false. Hence, the statement is not a tautology or a contradiction.

11. Show the equivalence of the following implications.

a) $P \Rightarrow Q$ (direct form of an implication)
b) $\sim Q \Rightarrow \sim P$ (contrapositive form)
c) $(P \wedge \sim Q) \Rightarrow \sim P$ (proof by contradiction)
d) $(P \wedge \sim Q) \Rightarrow Q$ (proof by contradiction)
e) $(P \wedge \sim Q) \Rightarrow R \wedge \sim R$ (proof by *reduction ad absurdum*)

Ans: Draw truth tables

12. **(Hmmmmmmmmmm)** Is the statement $(P \vee Q) \Leftrightarrow (P \vee \sim Q)$ true for all truth values of P and Q, or is it false for all values, or is it sometimes true and sometimes false?

Ans: It is true then P is true and Q is false, otherwise the statement is false as can be seen from the truth table.

			(1)	(2)	(3)	(4)
P	Q		$P \vee Q$	$\sim Q$	$P \vee \sim Q$	$(P \vee Q) \Leftrightarrow (P \vee \sim Q)$
T	T		**T**	F	**T**	**T**
T	F		**T**	T	**T**	**T**
F	T		**T**	F	**F**	**F**
F	F		**F**	T	**T**	**F**

13. **(Interesting Biconditional)** Is the statement $(P \vee Q) \Leftrightarrow (\sim P \vee \sim Q)$ true for all truth values of P and Q, or is it false for all values, or is it sometimes true and sometimes false?
Ans: It is true when exactly one of P and Q is true, and false when both are true or both are false as can be seen from the truth table.

			(1)	(2)	(3)	(4)	(5)
P	Q		$P \vee Q$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$	$(P \vee Q) \Leftrightarrow (\sim P \vee \sim Q)$
T	T		**T**	F	F	**F**	**F**
T	F		**T**	F	T	**T**	**T**
F	T		**T**	T	F	**T**	**T**
F	F		**F**	T	T	**T**	**F**

14. Find the negation of the following sentences.

a) $(P \vee Q) \wedge R$
Ans: $\sim [(P \vee Q) \wedge R] \equiv \sim (P \vee Q) \vee \sim R \equiv (\sim P \wedge \sim Q) \vee R$

b) $(P \vee Q) \wedge (R \vee S)$
Ans: $\sim [\sim Q \Rightarrow \sim P] \equiv \sim (Q \vee \sim P) \equiv \sim Q \wedge P$

c) $(\sim P \vee Q) \wedge R$
Ans: $\sim [(\sim P \vee Q) \wedge R] \equiv \sim (\sim P \vee Q) \vee \sim R \equiv (P \wedge \sim Q) \vee \sim R$

15. Give, if possible, an example of a true conditional sentence for which

a) the contrapositive is true
Ans: If a function f is differentiable then f is continuous.

b) the contrapositive is false
Ans: The contrapositive is equivalent to the conditional so this can never be.

c) the converse is true
Ans: If today is Monday, then it is the second day of the week. The converse of this sentence states that if this is the second day of the week, then it is Monday, which is also true. When both a conditional and its converse are true the hypothesis and conclusion are logically equivalent.

d) the converse is false

Ans:

If a function f is differentiable then f is continuous. Here the converse states that if a function is continuous, then it is differentiable, which is false. The function $f(x)=|x|$ is a counterexample of a continuous function that is not differentiable.

16. The **inverse** of the implication $P \Rightarrow Q$ is $\sim P \Rightarrow \sim Q$.

a) Prove or disprove that an implication and its inverse are equivalent.

Ans: Note that columns (1) and (4) are not the same. Hence the inverse of an

implication is not equivalent to the implication.

		(1)	(2)	(3)	(4)
P	Q	$P \Rightarrow Q$	$\sim P$	$\sim Q$	$\sim P \Rightarrow \sim Q$
T	T	**T**	F	F	**T**
T	F	**F**	F	T	**T**
F	T	**T**	T	F	**F**
F	F	**T**	T	T	**T**

b) What are the truth values of P and Q for which an implication and its inverse are both true?

Ans: When both P and Q are true or when they are both false.

c) What are the truth values of P and Q for which the implication and its inverse are both false?

Ans: They never have the same truth values.

17. For the sentence

"If N is an integer, then 2N is an even integer" write the converse, contrapositive, and inverse sentences.

Ans: **Converse:** If $2N$ is an even integer, then N is an integer.
Contrapositive: If $2N$ is an odd integer, then N is not an integer.
Inverse: If N is not an integer, then $2N$ is not an even integer.

18. Let P,Q, and R be sentences. Show

a) $P \Rightarrow (Q \Leftrightarrow R)$ requires paranthesis

Ans: One simply verifies that the truth tables for $P \Rightarrow (Q \Leftrightarrow R)$ and $(Q \Rightarrow P) \Leftrightarrow R$ are not identical.

b) $(P \wedge Q) \vee R$ requires paranthesis

Ans: One simply verifies that the truth tables for $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ are not identical.

c) $(\sim P \vee Q) \Rightarrow R$ may be written $\sim P \vee Q \Rightarrow R$

Ans: One simply verifies the truth tables for $(\sim P \vee Q) \Rightarrow R$ and $\sim P \vee Q \Rightarrow R$ are the same.

19. **(Challenge)** Rewrite the sentence

$$P \Rightarrow (Q \Rightarrow R)$$

in an equivalent form in which the symbol "$\Rightarrow$" does no occur.

Ans: Since $(Q \Rightarrow R) \equiv (\sim P \vee Q)$ we can write

$$[P \Rightarrow (Q \Rightarrow R)] \equiv \sim P \vee (Q \Rightarrow R) \equiv \sim P \vee (\sim Q \vee R)$$

20. **(Non Obvious Statement)** The statement

$$P \Rightarrow (Q \Rightarrow P)$$

can be read "If P is true, then P follows from any Q" Is this a tautology, contradiction, or does its truth value depend on the truth or falsity of P and Q?

Ans: Making a truth table for the sentence, we find

P	Q	$Q \Rightarrow P$	$P \Rightarrow (Q \Rightarrow P)$
T	T	T	**T**
T	F	T	**T**
F	T	F	**T**
F	F	T	**T**

Hence, the statement is a tautology.

21. **(Another Non Obvious Statement)** The statement

$$(Q \Rightarrow P) \vee (P \Rightarrow Q)$$

can be read "For any two sentences P and Q, it is always true that P implies Q or Q implies P" Is this a tautology, contradiction, or does its true value depend on the truth or falsity of P and Q?

Ans: Making a truth table for the sentence, we find

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \vee (Q \Rightarrow P)$
T	T	T	T	**T**
T	F	F	T	**T**
F	T	F	F	**T**
F	F	T	T	**T**

Hence, the statement is a tautology.

22. **(Three-Valued Logic)** Two-valued (T and F) truth tables were basic in logic until 1921 when the Polish logician Jan Lukasiewicz (1878-1956) and American logician Emil Post (1897-1954) introduced n-valued logical systems where n is any integer greater than 1. For example, sentences in a three-valued logic might have values True, False, and Unknown. Three-value logic is useful in computer science in database work. The truth tables for the AND, OR, and NOT connectives are as follows:

A	***B***	***A* OR *B***	***A* AND *B***	**NOT *A***
True	True	True	True	False
True	Unknown	True	Unknown	False
True	False	True	False	False
Unknown	True	True	Unknown	Unknown
Unknown	Unknown	Unknown	Unknown	Unknown
Unknown	False	Unknown	False	Unknown
False	True	True	False	True
False	Unknown	Unknown	False	True
False	False	False	False	True

From these connectives, derive the connectives for the conditional and biconditional connectives.

Ans: Using the fact that $P \Rightarrow Q \equiv \sim P \vee Q$, the conditional connective is

		(1)	(2)	(3)
P	Q	$\sim P$	$\sim P \vee Q$	$P \Rightarrow Q$
T	T	F	T	$\boldsymbol{T}$
T	U	F	U	$\boldsymbol{U}$
T	F	F	F	$\boldsymbol{F}$
U	T	U	T	$\boldsymbol{T}$
U	U	U	U	$\boldsymbol{U}$
U	F	U	U	$\boldsymbol{U}$
F	T	T	T	$\boldsymbol{T}$
F	U	T	T	$\boldsymbol{T}$
F	F	T	T	$\boldsymbol{T}$

Using the fact that p $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$ we have the biconditional in column (6)**.**

		(1)	(2)	(3)			(4) (5)	(6)
P	Q	$\sim P$	$\sim P \vee Q$	$P \Rightarrow Q$	$\sim Q$	$\sim Q \vee P$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	*T*	*F*	*T*	*T*	*F*	*T*	*T*	***T***
T	*U*	*F*	*U*	*U*	*U*	*T*	*T*	***U***
T	*F*	*F*	*F*	*F*	*T*	*T*	*T*	***F***
U	*T*	*U*	*T*	*T*	*F*	*U*	*U*	***U***
U	*U*	*U*	*U*	*U*	*U*	*U*	*U*	***U***
U	*F*	*U*	*U*	*U*	*T*	*T*	*T*	***U***
F	*T*	*T*	*T*	*T*	*F*	*F*	*F*	***F***
F	*U*	*T*	*T*	*T*	*U*	*U*	*U*	***U***
F	*F*	*T*	*T*	*T*	*T*	*T*	*T*	***T***

23. **(Modus Ponens and Modus Tollens)** **Modus Ponens**[1] and **Modus Tollens**[2] are systematic ways of making logical arguments that takes the form

If P then Q
P
Therefore Q

Modus Ponens

If P then Q
$\sim Q$
Therefore $\sim P$

Modus Tollens

Show that Modus Ponens and Modus Tollens are both tautologies,.

Ans: Modus Ponens is a tautology which can be seen by constructiong the truth table for $\left[(P \Rightarrow Q) \wedge P\right] \Rightarrow Q$ and observing there are all <u>*T*</u>'s in column (4) below:

		(1)	(2)	(3) (4)
P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge P$	$\left[(P \Rightarrow Q) \wedge P\right] \Rightarrow Q$
T	*T*	*T*	*T*	***T***
T	*F*	*F*	*F*	***T***
F	*T*	*T*	*F*	***T***
F	*F*	*T*	*F*	***T***

Modus Tollens is a tautology which can be seen by constructing the truth table for $\left[(P \Rightarrow Q) \wedge \sim Q\right] \Rightarrow \sim P$ and observing there are <u>*T*</u>'s in column (4) below:

[1] Latin: *mode that affirms.*
[2] Latin *mode that denies.*

		(1)	(2)		(3)	(4)
P	Q	$P \Rightarrow Q$	$\sim Q$	$(P \Rightarrow Q) \wedge \sim Q$	$\left[(P \Rightarrow Q) \wedge \sim Q\right] \Rightarrow \sim Q$	
T	*T*	*T*	*F*	*F*	***T***	
T	*F*	*F*	*T*	*F*	***T***	
F	*T*	*T*	*F*	*F*	***T***	
F	*F*	*T*	*T*	*T*	***T***	

24. **(Interesting)** Are the following two statements equivalent?

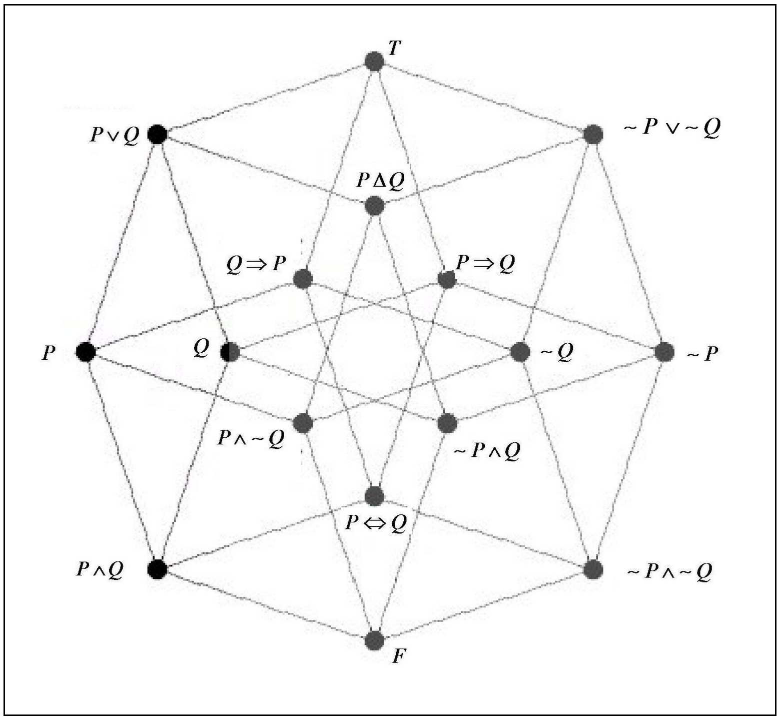
$$P \wedge (Q \Rightarrow R)$$

$$(P \wedge Q) \Rightarrow R$$

Ans: Yes, as can be seen from the following truth table.

			(1)	(2)	(3)	(4)
P	Q	R	$Q \Rightarrow R$	$P \wedge Q$	$P \wedge (Q \Rightarrow R)$	$(P \wedge Q) \Rightarrow R$
T	T	T	T	T	**T**	**T**
T	T	F	F	T	**F**	**F**
T	F	T	T	F	**T**	**T**
T	F	F	T	F	**T**	**T**
F	T	T	T	F	**F**	**T**
F	T	F	F	F	**F**	**T**
F	F	T	T	F	**F**	**T**
F	F	F	T	F	**F**	**T**

25. **(16 Logical Functions of Two Variables)** The diagram below shows the logical relations between the 16 logical expressions of two logical variables. One expression can be proven from another if it lies on an upward path from the first. For example $(P \wedge Q) \Rightarrow Q \Rightarrow (P \Rightarrow Q)$. Verify a few of these implications using truth tables. The compound sentence $P \Delta Q$ refers to the exclusive OR, which means either P or Q is true but not both.



Ans: Straight forward use of truth tables.

26. **(Professor Snarf's Birthday)** Professor Snarf tells Mary the *month* of his birthday and tells Dave the *day* of his birthday, and tells them not to pass this information to the other. Professor Snarf then tells them if they can deduce his birthday, they will each receive an A in their logic course, knowing it is impossible. However, Professor Snarf does not realize how smart Mary and Dave are in logic.

		Day					
		1	2	3	4	5	6
Montth	Jan			Jan 3	Jan 4		Jan 6
	Feb			Feb 3		Feb 5	
	March	March 1			March 4		
	April	April 1	April 2				April 6

After a moment's thought:

Statement 1: Mary says, "If I don't know the answer, then Dave doesn't know either.
Statement 2: Dave says "I didn't know the answer before, but now I know."

Statement 3: Mary says: "I now know the answer too."

whereupon they correctly tell Professor Snarf his birthday and he grudgenly gives them their As. What was the correct answer that Mary and Dave gave Professor Snarf?
Ans: Let us rename Mary by MONTH and Dave by DAY to so we don't forget that MONTH knows the month and DAY knows the day.

In a nutshell, MONTH is going to pass some valuable information to DAY (enough so DAY knows the birthday), and then DAY passes back some information to MONTH (enough so that MONTH knows the birthday), and then MONTH passes you the reader information so YOU know the birthday.

At the start MONTH (who knows the month but not the day) does not know the birthday and so Statement 1, which is an implication of the form $P \Rightarrow Q$ implies that neither does DAY This is MONTH's clever day of telling DAY that the month is not February or April, since if the birthday *were* in one of those two months, DAY would know inasmuch as day 2 only occurs in April and day 5 only in Feb. So now MONTH and DAY reduced the number of possible birthdays from 10 to 5 as illustrated below.

		Day					
		1	2	3	4	5	6
Montth	Jan			Jan 3	Jan 4		Jan 6
	March	March 1			March 4		

It is now DAY's turn to deduce the answer and pass enough information back to MONTH so she can deduce the answer. DAY says he *knows* the birthday, which tells us MONTH it is not on the 4^{th} of the month. (It is easy for him to find the answer, he just looks in the column of the day Professor Snarf gave him to find the month. So now we are down to three possibilities March 1, Jan 3, or Jan 6.

		Day					
		1	2	3	4	5	6
Montth	Jan			Jan 3			Jan 6
	March	March 1					

Now, it is MONTH’s turn to deduce the birthday and tell you ! She says she knows the birthday, which means the birthday must by March 1, otherwise if her month was Jan she would be unable to decide between Jan 3 and Jan 6. Hence, Professor Snarf’s birthday is March 1.

ΦΜΠΟΞϒ