Section 1.3 Predicate Logic

1. Write the following famous quotes in the symbolic language of predicate logic.

a) Learn from yesterday, live for today, hope for tomorrow.

....Albert Einstein

Ans:

$(\forall \text{ yesterdays } y)(\forall \text{ todays } t)(\forall \text{ tomorrows } m)(\text{learn from } y,\ \text{live for } t,\ \text{hope for } m)$

b) A woman can say more in a sigh than a man can say in a sermon.

...Arnold Haultain

Ans: $(\forall \text{ woman } w)(\forall \text{ man } m)(w \text{ says more with a sign than does } m \text{ in a sermon})$

c) "We all go a little mad sometimes."

...Norman Bates in Psycho (1960)

Ans: $(\forall \text{ humans } h)(\exists \text{ time } t)(h \text{ goes mad at time } t)$

d) Cowards die many times before their deaths; the valiant never taste of death but once.

... William Shakespeare

Ans:

$(\forall \text{ cowards } c)(\forall \text{ valient persons } v)(c \text{ die many times before death but } v \text{ taste of death but once})$

e) All people are mortal.

Ans: $(\forall x)(x \text{ is a person} \Rightarrow x \text{ is mortal})$

f) All that glitters is not gold.

Ans: $\sim(\forall \text{things})(\text{a thing that glitter } \Rightarrow \text{ the thing is gold})$

The answer is not

$$(\forall \text{things})(\text{a thing that glitter } \Rightarrow \text{ the thing is not gold})$$

2. Write the following sentences in the symbolic language of predicate logic. The universe of each variable is given in parenthesis. For these problems we use the notation

$\mathbb{Z} =$ integers

$\mathbb{R} =$ real numbers

a) If $a|b$ and $b|c$, then $a|c$, where a,b,c are integers. (Integers)

Ans: $(\forall a,b,c \in \mathbb{Z})\left[a \mid b \wedge b \mid c \Rightarrow a \mid c\right]$

b) 4 does not divide n^2+2 for any integer (Integers)
Ans: $(\forall n \in \mathbb{Z})(\sim \exists m \in \mathbb{Z})(n^2+2=4m)$ or
$(\forall n \in \mathbb{Z})(\nexists m \in \mathbb{Z})(n^2+2=4m)$ where $\nexists$ denotes there does not exist.

c) $x^3+x+1=0$ for some real x (Real numbers)
Ans: $(\exists x \in \mathbb{R})(x^3+x+1=0)$

d) Everybody loves mathematics. (All people)
Ans: $(\forall p \in P)(p \text{ loves math})$

e) For every positive real number a there exists a real number x that satisfies $e^x=a$. (Real numbers)
Ans: $(\forall a \in \mathbb{R}_+)(\exists x \in \mathbb{R})(e^x=a)$

f) For every positive real number $\varepsilon>0$ there exists a real number $\delta>0$ such that $|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$ where a,x are arbitrary real numbers.

Ans: $(\forall \varepsilon \in \mathbb{R}_+)(\exists \delta \in \mathbb{R}_+)(\forall a,x \in \mathbb{R})\left(|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon\right)$ More often one assumes that $\varepsilon,\delta,x,$ and a are real numbers and simply writes
$(\forall \varepsilon>0)(\exists \delta>0)\left(|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon\right)$

g) Everyone always attends class. (All students)
Ans: $(\forall s \in S)(s \text{ attends class})$

h) The equation $x^2+1=0$ has no solution (Real numbers)
Ans: $(\sim \exists x \in \mathbb{R})(x^2+1=0)$ **or** $(\nexists x \in \mathbb{R})(x^2+1=0)$

h) The equation $x^2-2=0$ has no solution (Rational numbers)
Ans: $(\sim \exists x \in \mathbb{Q})(x^2-2=0)$ or $(\nexists x \in \mathbb{Q})(x^2-2=0)$

3. **(True or False?)** Which of the following are true in the given universe? The universe is given in parenthesis.

a) $(\forall x)(x \le x)$ (Real numbers) **Ans:** T
b) $(\exists x)(x^2=2)$ (Real numbers) **Ans:** T

c) $(\exists x)(x^2 = 2)$ (Rational numbers) **Ans:** F
d) $(\exists x)(x^2 + x + 1 = 0)$ (Real numbers) **Ans:** T
e) $(\forall x)[x \equiv 1(\bmod 5)]$ (Integers) **Ans:** T
f) $(\exists ! x)(e^x = 1)$ (Real numbers) **Ans:** T
g) $(\forall x)(x \le x)$ (Integers) **Ans:** T

4. **(True or False**) Very quickly, true or false. Draw Venn diagrams to convince yourself of your answers.

a) $\sim (\forall x)(P(x)) \equiv (\exists x)[\sim P(x)]$ **Ans:** T
b) $\sim (\exists x)(P(x)) \equiv (\forall x)[\sim P(x)]$ **Ans:** T
c) $\sim (\forall x)[\sim P(x)] \equiv (\exists x)[P(x)]$ **Ans:** T
d) $\sim (\exists x)[\sim P(x)] \equiv (\forall x)[P(x)]$ **Ans:** T
e) $(\forall x)[P(x) \Rightarrow Q(x)] \equiv \sim (\exists x)[P(x) \wedge \sim Q(x)]$ **Ans:** T
f) $\sim (\exists x)[P(x) \wedge Q(x)] \equiv (\forall x)[P(x) \Rightarrow \sim Q(x)]$ **Ans:** T

5**. (Expanding Universes)** In which universes $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are the following sentences true?

a) For every x in the universe there exists a $y = 1 - x$ in the universe. **Ans:** $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

b) For every $x \ne 0$ in the universe there exists a $1/x$ in the universe. **Ans:** $\mathbb{Q}, \mathbb{R}, \mathbb{C}$

c) For every x in the universe there exists a solution of $x^2 - 2 = 0$ in the universe. **Ans:** $\mathbb{R}, \mathbb{C}$

d) For every x in the universe there exists a solution of $x^2 + 1 = 0$ in the universe. **Ans:** $\mathbb{C}$

6. **(Not as Easy as It Looks)** Tell if $(\exists x \in U)[x \text{ is even } \Rightarrow 5 \le x \le 10]$ is true or false for the given universe U.

a) $U = \{4\}$ **Ans:** F

b) $U = \{3\}$ **Ans:** T

c) $U = \{6, 8, 10\}$ **Ans:** T

d) $U = \{6,8,10,12\}$ **Ans:** T

e) $U = \{6,7,8,10,12\}$ **Ans:** T

7. **(Small Universe)** Which of the following statements are true for the universe $U = \{1,2,3\}$.

a) $1 < 0 \Rightarrow (\exists x)(x < 0)$ **Ans:** T

b) $(\exists x)(\forall y)(x \le y)$ **Ans:** F

c) $(\forall x)(\exists y)(x \le y)$ **Ans:** T

d) $(\exists x)(\exists y)(y = x + 1)$ **Ans:** F

e) $(\forall x)(\forall y)(xy = yx)$ **Ans:** T

f) $(\forall x)(\exists y)(y \le x + 1)$ **Ans:** T

8. **(Well-Known Universe)** Letting

$R(x)$: x is a rational number

$I(x)$: x is an irrational number

which of the following statements are true for the universe of real numbers.

a) $(\forall x)[I(x) \vee R(x)]$ **Ans:** T

b) $(\forall x)[I(x) \wedge R(x)]$ **Ans:** F

c) $(\forall x)R(x) \vee (\forall x)I(x)$ **Ans:** F

d) $(\forall x)[R(x) \vee I(x)] \Rightarrow [(\forall x)R(x) \vee (\forall x)I(x)]$ **Ans:** F

e) $[(\forall x)R(x) \vee (\forall x)I(x)] \Rightarrow (\forall x)[R(x) \vee I(x)]$ **Ans:** T

9. **(Famous Theorems)** State the following famous theorems in mathematics in the symbolic language of predicate logic.

a) **(Bolzano's Intermediate Value Theorem)** If f is a continuous function on an interval $[a,b]$ and if f changes sign from negative to positive (or vice versa), then there exists a c between a and b such that $f(c) = 0$.

Ans: Let $[a,b]$ be an interval on the real line and $f \in C[a,b]$. Then f changes sign from positive to negative (or vise versa)$\Rightarrow (\exists c \in [a,b])(f(c) = 0)$. One normally does not quantify the function f in symbolic form, as in $(\forall f \in C[a,b])$, since predicate logic only quantifies *variables* (like integers, rational numbers, real numbers, etc) and not functions.

b) **(Fermat's Last Theorem)** If n is an integer greater than 2, then there are no nonzero integer values of a, b, c that satisfy $a^n + b^n = c^n$.
Ans: $(\forall n \in \mathbb{Z})\left[n > 2 \Rightarrow (\nexists a, b, c \neq 0) \text{ such that } a^n + b^n = c^n\right]$

c) **(Euler's Theorem)** If P is any regular polyhedra, and v, e, f represent the number of ertices, edges, and faces, respectively of the polyhedra, then $v - e + f = 2$.
Ans: Let P be any regular polyhedra. Then

v, e, f are the vertices, edges, and faces of $P \Rightarrow v - e + f = 2$

d) **Binomial Theorem)** If a, b are real numbers and n is a nonnegative integer, then

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}$$

Ans: $(\forall a, b, c \in \mathbb{R})(\forall n \in \mathbb{Z})\left(n > 0 \Rightarrow (a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}\right)$

10. **(Negation)** Negate the following sentences in words

a) All women are moral.
Ans: There exists a woman who is not moral;

b) Every player on the basketball team was over six feet tall.
Ans: There is a player on the team who is not over six feet tall.

c) For any real number y there exists a real number x that satisfies $y = \tan x$.
Ans: There exists a real number y such that for all real numbers x one has $y \neq \tan x$.

d) The exists a real number x such that $0 < x < 5$ and $x^3 - 8 = 0$.
Ans: For every real number x, one has $x \notin (0,5)$ or $x^3 - 8 \neq 0$

e) The equation $a^n + b^n = c^n$ does not have nonzero integer solutions a, b, c for n a natural number $n > 2$.
Ans: There exists a natural number $n > 2$ that has solutions of the equation $a^n + b^n = c^n$ for nonzero integers a, b, c.

11. **(Negation in Predicate Logic)** Negate the following sentences in symbolic form.

f) $(\forall x)[P(x) \Rightarrow Q(x)]$

Ans: $(\exists x)\left[P(x) \not\Rightarrow Q(x)\right] \equiv (\exists x)\left[P(x) \wedge \sim Q(x)\right]$

g) $(\forall x)\left[x > 0 \Rightarrow (\exists y)(x^2 = y)\right]$

Ans:

$(\exists x)\left[x > 0 \not\Rightarrow (\exists y)(x^2 = y)\right]$

$\equiv (\exists x)\left[(x > 0) \wedge \sim (\exists y)(x^2 = y)\right]$

$\equiv (\exists x)\left[(x > 0) \wedge (\forall y)(x^2 \neq y)\right]$

h) $P \Rightarrow (Q \wedge R)$

Ans: $\sim\left[P \Rightarrow (Q \wedge R)\right] \equiv \sim\left[\sim P \vee (Q \wedge R)\right] \equiv \left[P \wedge \sim (Q \wedge R)\right] \equiv P \wedge (\sim Q \vee \sim R)$

i) $(P \wedge Q) \Rightarrow R$

Ans:

$\sim\left[(P \wedge Q) \Rightarrow R\right] \equiv \sim\left[\sim (P \wedge Q) \vee R\right]$

$\equiv \sim\left[(\sim P \vee \sim Q) \vee R\right]$

$\equiv \sim (\sim P \vee \sim Q) \wedge \sim R$

$\equiv (P \wedge Q) \wedge \sim R$

$\equiv P \wedge Q \wedge \sim R$

12. **(Convergence and Non-convergence)** A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges to a
limit L if and only if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|x_n - L| < \varepsilon)$$

a) State the negation of this sentence.

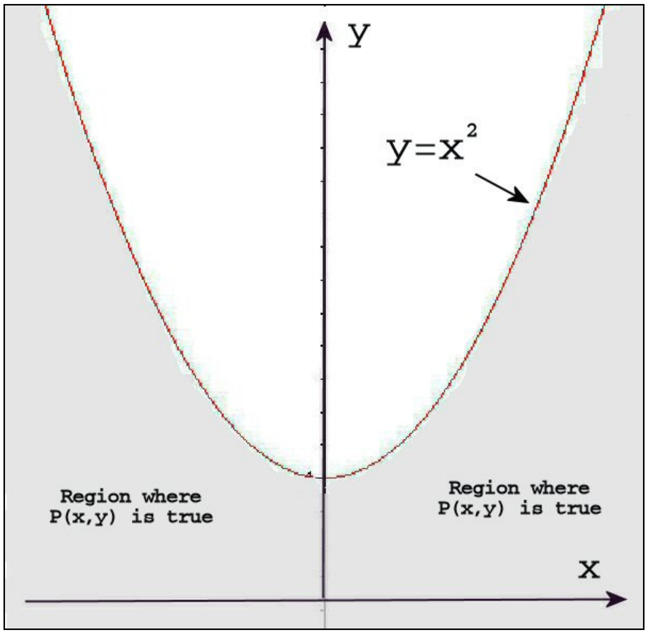
b) Using the negation found in a) prove that the sequence $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ does not converge to $L = 1/4$.

Ans:

a) $(\exists \varepsilon > 0)(\forall N \in \mathbb{N})(\exists n > N)\left(|x_n - L| \geq \varepsilon\right)$

b) Pick $\varepsilon = 1/20$. Then for any natural number $N = 1, 2, \ldots$ we can pick $n = N + 5$, so for $N = 1, 2, \ldots$ the corresponding values of n will be $n = 6, 7, 8$ which are all farther from $L = 1/4$ than $\varepsilon = 1/20$.

13. **(Graph to the Rescue)** Starting with the universe of points (x, y) in the Cartesian plane $\mathbb{R}^2$ and consider the points that satisfy $P(x, y) : y \le x^2 + 1$, represented by the shaded points in following figure.



Truth Values of $P(x, y)$

Determine which of the following sentences are true.

a) $(\forall x)(\forall y)\, P(x, y)$ **Ans:** F
b) $(\forall x)(\exists y)\, P(x, y)$ **Ans:** T
c) $(\exists x)(\forall y)\, P(x, y)$ **Ans:** F
d) $(\exists x)(\exists y)\, P(x, y)$ **Ans:** T
e) $(\exists y)(\forall x)\, P(x, y)$ **Ans:** T
f) $(\forall y)(\exists x)\, P(x, y)$ **Ans:** T

14. **(Order Counts)** Which of the following are true and which are false for real numbers x, y?

a) $(\forall x)(\forall y)(x < y)$ **Ans:** F
b) $(\exists x)(\forall y)(x < y)$ **Ans:** F

c) $(\forall y)(\exists x)(x < y)$ **Ans:** T

d) $(\exists x)(\exists y)(x < y)$ **Ans:** T

15. **(Fun Time)** State the denial of the words of wisdom attributed to Abraham Lincoln: "You can fool some of the people, all the time and all the people some of the time, but you can't fool all the people all of the time."

16. **(In Plain English)** Restate the following sentences in plain English.

a) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\big[(x < y) \Rightarrow (\exists z \in \mathbb{R})\big[(x < z) \wedge (z < y)\big]\big]$

Ans: For all real numbers x, y if $x < y$ then there exists a real number z such that $x < z$ and $z < y$.

b) $(\forall x \in \mathbb{R})\big[(x > 0) \Rightarrow (\exists y \in \mathbb{R})(x = y^2)\big]$

Ans: For any positive real number x there exists a real number y such that $x = y^2$.

c) $(\forall m, n \in \mathbb{N})\big[(n > 1) \Rightarrow \big[m \mid n \Rightarrow (m = 1) \vee (m = n)\big]\big]$ ($m \mid n$ means m divides n)

Ans: For all natural numbers m, n where $n > 1$, if m divides n, then either $m = 1$ or $m = n$.

d) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\big[(x < y) \Rightarrow (\exists z \in \mathbb{R})\big[(x < z) \wedge (z < y)\big]\big]$

Ans: For all real numbers x, y if $x < y$, then there exists a real number z that satisfies $x < z$ and $z < y$.

17. **(True for You Might be False for Me)** Tell if the following are true or false in the universes $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

a) $(\forall x)(\exists y)(x < y)$

Ans: Universe $\mathbb{N}$: T, Universe $\mathbb{Z}$:T , Universe $\mathbb{Q}$:T , Universe $\mathbb{R}$:T

b) $(\forall y)(\exists x)(x < y)$

Ans: Universe $\mathbb{N}$: F, Universe $\mathbb{Z}$:T , Universe $\mathbb{Q}$:T , Universe $\mathbb{R}$:T

c) $(\exists x)(\forall y)(x < y)$

Ans: Universe $\mathbb{N}$: F, Universe $\mathbb{Z}$:F , Universe $\mathbb{Q}$: F , Universe $\mathbb{R}$:F

d) $(\exists y)(\forall x)(x < y)$

Ans: Universe $\mathbb{N}$: F, Universe $\mathbb{Z}$: F , Universe $\mathbb{Q}$:F , Universe $\mathbb{R}$:F

e) $(\forall x)\left[(x>0)\Rightarrow(\exists y)\left(y=x^2\right)\right]$

Ans: Universe $\mathbb{N}$: T, Universe $\mathbb{Z}$:T , Universe $\mathbb{Q}$:T , Universe $\mathbb{R}$:T

18. **(Satisfiable Sentences in Predicate Logic)** A **satisfiable** sentence in predicate logic is a sentence that is true under *some* universe of individual values. For example, the sentence $(\exists x\in U)(x>0)$ is satisfiable since it is true for $x\in U$ in the universe of real numbers. Tell if the following sentences are satisfiable.

a) $(\exists x\in U)\left[(x>0)\wedge(x<0)\right]$ **Ans:** NO

b) $(\exists x\in U)\left(x^2=-3\right)$ **Ans:** YES: $U=\mathbb{C}$

c) $(\exists x\in U)\left[P(x)\wedge\sim P(x)\right]$ **Ans:** NO

d) $(\exists x\in U)\left[(x>0)\vee(x<0)\right]$ **Ans:** YES: $U=\mathbb{R}$

19. **(Translation into Predicate Logic)** Letting

$$E(x)=x \text{ is even}$$
$$O(x)=x \text{ is odd}$$

translate the following sentence to predicate logic.

a) Not every integer is even.
Ans: $(\exists x\in\mathbb{Z})\left[x\notin E(x)\right]$

b) Some integers are odd.
Ans: $(\exists x\in\mathbb{Z})\left[x\in O(x)\right]$

c) Some integers are even and some integers are odd.
Ans: $(\exists x\in\mathbb{Z})\left[x\in E(x)\right]\wedge(\exists x\in\mathbb{Z})\left[x\in O(x)\right]$

d) If an integer is even then it is not odd.
Ans: $(\forall x\in\mathbb{Z})\left[x\in E(x)\Rightarrow x\notin O(x)\right]$

e) If an integer is even then two larger than the integer is even.
Ans: $(\forall x\in\mathbb{Z})\left[x\in E(x)\Rightarrow x+2\in E(x)\right]$

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