## Section 1.5, Proofs in Predicate Logic

1.. **(True or False?)** Which of the following are true?

a) $(\forall x\in\mathbb{R})\left(x^2+x+1>0\right)$ **Ans:** T

b) $(\forall x\in\mathbb{R})\left[x^2>0\vee x^2<0\right]$ **Ans:** F

c) $(\forall x\in\mathbb{Z})\left(x^2>x\right)$ **Ans:** F

d) $(\forall x\in\mathbb{R})(\exists y\in\mathbb{R})(y=\sin x)$ **Ans:** F

e) $(\forall x\in\mathbb{R})(\exists y\in\mathbb{R})(y=\tan x)$ **Ans:** T

f) $(\exists x\in\mathbb{R})(\exists y\in\mathbb{R})(y=\sin x)$ **Ans:** F

g) $(\exists x,y\in\mathbb{N})(\exists n>2)\left(x^n+y^n=1\right)$ **Ans:** F

h) $(\exists x\in\mathbb{R})(\forall a,b,c\in\mathbb{R})\left(ax^2+bx+c=0\right)$ **Ans:** T

i) $(\exists x\in\mathbb{C})(\forall a,b,c\in\mathbb{R})\left(ax^2+bx+c=0\right)$ **Ans:** T

j) $(\forall\varepsilon>0)(\exists N\in\mathbb{N})(\forall n>N)\left(\frac{1}{n}<\varepsilon\right)$ **Ans:** T

k) $(\forall\varepsilon>0)(\exists\delta>0)\left(|x-2|<\delta\Rightarrow\left|x^2-4\right|<\varepsilon\right)$ **Ans:** T

2. **(Predicate Logic Form)** Write the following theorems as predicate logic notation.

a) A number is divisible by 4 if and only if its last two digits are.
**Ans**: $(\forall n\in\mathbb{N})\left(4\,|\,n\Leftrightarrow 4\,|\ \text{last two digits of } n\right)$

b) A natural number is divisible by $2^n$ if and only if its last $n$ digits are.
**Ans:** $(\forall m\in\mathbb{N})\left(2^n\,|\,m\Leftrightarrow 2^n\,|\ \text{last } n \text{ digits of } m\right)$

c) There exists irrational numbers $x,y$ such that $x^y$ is rataional.
**Ans:** $(\exists x,y\in\mathbb{R}-\mathbb{Q})\left(x^y\in\mathbb{Q}\right)$

d) $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$

**Ans:** $(\forall n\in\mathbb{N})\left(\sum_{k=1}^{n}k=\frac{n(n+1)}{2}\right)$

e) For positive real numbers $a,b$, we have $\sqrt{ab}\le\frac{a+b}{2}$.

**Ans:** $(\forall a,b\in\mathbb{R})\left(a>0,b>0\Rightarrow\sqrt{ab}\le\frac{a+b}{2}\right)$

f) If $a,b$ are integers and $b \neq 0$ then there exists a unique integers $q,r$ such that $a = qb + r$ where $0 \le r < |b|$.

**Ans:** $(\forall a,b \in \mathbb{Z})\left[b \neq 0 \Rightarrow (\exists! q \in \mathbb{Z})(\exists! r, 0 \le r < |b|)(a = qb + r)\right]$

g) If $p$ is a prime number that does not divide the integer $a$ then $p$ divides $a^p - a$.

**Ans:** $(\forall a \in \mathbb{Z})(\forall p \in \mathbb{N},\ p \text{ prime})(p \nmid a \Rightarrow p \mid a^p - a)$ **Note:** Prime numbers are positive integers greater than 1 than have factors only themselves and 1. By definition, negative integers are not considered prime numbers.

h) The square of any natural number must have each prime number occurring an even number of times in the prime factorization of the number.

**Ans:** $(\forall n \in \mathbb{N})(\text{each prime factor of } n^2 \text{ occurs an even number of times})$

i) All prime numbers $p$ greater than 2 are either of the form $p = 2n + 1$ or $p = 2n + 3$ for some natural number $k$.

**Ans:** $(\forall p \in \{\text{prime numbers}\})\left[p > 2 \Rightarrow (p = 2n + 1) \vee (p = 2n + 3)\right]$

j) Every even number greater than 4 can be expressed as the sum or two prime numbers.

**Ans:** $(\forall n \text{ an even integer})(n > 4 \Rightarrow n \text{ is the sum of two primes})$

a) **(Euler's Conjecture)** There are no natural numbers $a,b,c,d$ that satisfy the equation $a^4 + b^4 + c^4 = d^4$.

**Ans:** $(\nexists a,b,c,d \in \mathbb{N})(a^4 + b^4 + c^4 = d^4)$**.** **Note:** This conjecture is not a theorem since it was disproven in 1988 by Noam Elkies with the counterexample

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4.$$

3. **(Negation)** Negate the following theorems. Which is true, the original statement or its negation? Let the universe of all variables be the real numbers.

a) Original statement: $(\exists x)(\forall y)(xy < 1)$

**Ans:** Negation: $(\forall x)(\exists y)(xy \ge 1)$ The original statement is true and the negation false. The negation is false since if $x = 0$ there does not exist a real number $y$ that makes $xy \ge 1$.

b) Original: $(\forall x)(\forall y)(\exists z)(xyz = 1)$

**Ans:** Negation: $(\exists x)(\exists y)(\forall z)(xyz \neq 1)$ The negation is true. The original statement is false

since if $x = 0$ or $y = 0$, there does not exist a real number $z$ that makes $xyz = 1$.

c) Original: $(\forall x)(\forall y)(\forall z)(\exists w)(x^2 + y^2 + z^2 + w^2 = 0)$

**Ans:** Negation: $(\exists x)(\exists y)(\exists z)(\forall w)(x^2 + y^2 + z^2 + w^2 \neq 0)$ The negation is true and the original statement is false. The original statement is false since if $x = y = z = 1$, then there does not exist a real number $w$ that satisfies the equation $x^2 + y^2 + z^2 + w^2 = 0$.

d) Original: $\sim(\exists x)(\forall y)(x < y)$

**Ans:** Negation: $(\exists x)(\forall y)(x < y)$ The original is true and the negation is false. The negation is false since no matter how we pick $x$ we can pick $y$ less than $x$.

e) Original: $(\forall x)(\exists y)(xy < 1 \wedge xy > 1)$

**Ans:** Negation: $(\exists x)(\forall y)(xy \geq 1 \vee xy \leq 1)$ The negation is false since no matter what value of $x$ we can pick $y = 0$ and it is not true that $xy \geq 1 \vee xy \leq 1$.

f) Original: $(\exists x)(\exists y)(xy = 0 \vee xy \neq 0)$

**Ans:** Negation: $(\forall x)(\forall y)(xy \neq 0 \wedge xy = 0)$ The original is true and the negation false. The negation is not true since if $x = y = 1$ is a counterexample.

4. **(Counterexamples)** All the statements in this problem are wrong. Your job is to prove them wrong by finding a counterexample[1]. Counterexamples in mathematics tell us we are going in the wrong direction.

a) All mathematicians make tons of money writing textbooks.

**Ans:** Look no further than the book you are reading.

b) For all positive integers, $x^2 + x + 41$ is prime y[2].

**Ans:** Let $x = 41$. Hence $f(41) = 41^2 + 41 + 41 = 41 \cdot (41 + 1 + 1) = 41 \cdot 43$

c) Every continuous function defined on the interval (0,1) has a maximum and minimum value.

**Ans:** The function $f(x) = x$ is continuous on $(0,1)$ but has no maximum or minimum value in the interval $(0,1)$.

---

[1] A nice book outlining many counterexamples in mathematics is *Counterexamples in Mathematics* by Bernard R. Gelbaum and John Olmsted (Holden Day, Inc), 1964.

[2] A famous mathematician once said that he was once $x^2$ years old in the year $x$. Can you determine the year this person was born?

d) Every continuous function is differentiable.
**Ans:** The function $f(x)=|x|$ defined on $(-1,1)$, is continuous but not differentiable since it has no derivative when $x=0$

e) If the terms of an infinite series approach zero, then the series converges.
**Ans:** The terms in the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$ approach zero but the series diverges.

f) The perimeter of a rectangle can never be an odd integer.
**Ans:** Let the length of a rectangle by $x=5.25$ and the height by $y=7.25$. The perimeter is $2(5.25)+2(7.25)=25\,25$.

g) $(\forall x\in\mathbb{R})(\exists y\in\mathbb{R})(x=y^2)$
**Ans:** If $x=-1$ there is no real number $y$ that satisfies $x=y^2$

h) Every even natural number is the sum of two primes.
**Ans:** he even natural number $n=2$ is not the sum of two primes. Note: However is a conjecture, called the Goldbach Conjecture, that every even natural number greater than 2 is the sum of two primes. For example $4=2+2$, $6=3+3$, $8=5+3$, $10=7+3$, and so on. This conjecture has never been proven to be true, 300 years since the conjecture was first stated.

i) If $m$ and $n$ are positive integers such that $m$ divides $n^2-1$, then $m$ divides $n-1$ or $m$ divides $n+1$.
**Ans:** If $m=8, n=5$, then $m=8$ divides $n^2-1=24$ although $m=8$ does not divide either $n-1=4$ or $n+1=6$.

j) If $\{f_n : n=1,2,...\}$ is a sequence of continuous functions defined on $(0,1)$ that converge to a function $f$, then $f$ is continuous.
**Ans:** Each function in the sequence $f_n(x)=\sin^n(x), x\in[0,\pi]$ is continuous, but the limiting function is

$$f(x)=\lim_{n\to\infty} f_n(x)=\begin{cases}0 & x\neq\pi/2\\ 1 & x=\pi/2\end{cases}\text{s}$$

Is discontinuous. The reader will learn in a basic real analysis class that in order for the limit of a sequence of continuous functions to be continuous, the convergence must be *uniform* convergence, which in this counterexample, convergence is only pointwise convergence.

5. **(If and Only If Theorems)** State each of the following "if and only if" theorems in symbolic predicate logic notation. In each case we assume $f$ is a real valued function of a real variable.

a) A function $f$ is even iff for every real number $x$, $f(x) = f(-x)$.

**Ans:** $(\forall x \in \mathbb{R})(f \text{ even} \Leftrightarrow f(x) = f(-x))$

b) A function $f$ is odd iff for every real number $x$, $f(x) = -f(-x)$.

**Ans:** $(\forall x \in \mathbb{R})(f \text{ odd} \Leftrightarrow f(x) = -f(-x))$

c) A function $f$ is periodic iff there exists a $p$ such that $f(x) = f(x+p)$ for all real numbers $x$.

**Ans:** $(f \text{ periodic}) \Leftrightarrow (\forall x \in \mathbb{R})(\exists p \in \mathbb{R})[f(x) = f(x+p)]$

d) A function is increasing iff for every real numbers $x$ and $y$ we have $x \le y \Rightarrow f(x) \le f(y)$.

**Ans:** $(f \text{ increasing}) \Leftrightarrow (\forall x, y \in \mathbb{R})[x \le y \Rightarrow f(x) \le f(y)]$

e) A function $f$ is continuous at $x_0$ iff for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

**Ans:** $f$ continuous at $x_0 \Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0)\big[|x-y| < \delta \Rightarrow |f(x) - f|y|| < \varepsilon\big]$

f) A function $f$ is uniformly continuous on a set $E$ iff for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for any $x, y$ in $E$ that satisfy $|x - y| < \delta$..

**Ans:**

$f$ uniformly continuous on $E \Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in E)\big[|x-y| < \delta \Rightarrow |f(x) - f|y|| < \varepsilon\big]$

6. **(Hard or Easy?)** Assume there exists $a \in \mathbb{R}$ such that $a^3 + a + 1 = 0$. Prove there exists a $b \in \mathbb{R}$ such that $b^3 + b - 1 = 0$.

**Ans**: Let $b = -a$. Then

$$\begin{aligned} b^3 + b + 1 &= (-a)^3 + (-a) - 1 \\ &= -a^3 - a - 1 \\ &= -(a^3 + a + 1) \\ &= 0 \end{aligned}$$

7. **(Predicate Logic in Analysis)** The so-called $\varepsilon-\delta$ proofs in analysis were originated by Weierstrass in the late 1800s. They involve inequalities and a universal and existential quantifier. They normally start with $(\forall\varepsilon>0)$ which you might interpret as saying for any positive number $\varepsilon$ your worst enemy gives you (think very small), you can find a corresponding positive real number $\delta>0$ that (hopefully) satisfies a given relation between $\varepsilon$ and $\delta$. The idea is that your adversary can pick $\varepsilon>0$ as small as he or she pleases, but you have the advantage of picking the $\delta$ second. Of course, your choice of $\delta$ will most likely depend on $\varepsilon$ (i.e. be a function of $\varepsilon$).

a) Show that for every real number $\varepsilon>0$ there exists a real number $\delta>0$ such that $2\delta<\varepsilon$. In the language of predicate logic, prove

$$(\forall\varepsilon>0)(\exists\delta>0)(2\delta<\varepsilon).$$

**Ans:** For every real number $\varepsilon>0$, we only need show the existence of one $\delta>0$ that satisfies $2\delta<\varepsilon$, which we can do by picking $\delta=\varepsilon/4$ since $2\delta=2(\varepsilon/4)=\varepsilon/2<\varepsilon$. Note that $\varepsilon$ was an arbitrary positive real number and that we only had to find *one* $\delta>0$ that satisfied the given condition.

b) For every real number $\varepsilon>0$, there exists an integer $N>0$ such that for $n>N$ one has $1/n<\varepsilon$. In the language of predicate logic, prove

$$(\forall\varepsilon>0)(\exists N>0)(\forall n>N)(1/n<\varepsilon).$$

**Ans:** If we pick $N=\varepsilon$, then for any $\varepsilon>0$ and $n>N$ one has $1/n<1/N=\varepsilon$. Congratulations you just proved $\lim_{n\to\infty}(1/n)=0$.

c) Show that for every real number $\varepsilon>0$, there exists a real number $\delta>0$ such that if $|x|<\delta$ then $x^2<\varepsilon$. In the language of predicate logic, prove

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall x\in\mathbb{R})\left(|x|<\delta\Rightarrow x^2<\varepsilon\right).$$

**Ans:** If $\delta=\sqrt{\varepsilon}$, then $|x|<\delta=\sqrt{\varepsilon}\Rightarrow x^2<\varepsilon$. Congratulations you just proved that the function $f(x)=x^2$ is continuous at $x=0$.

d) Show that for every positive integer $M$ there is a positive integer $N$ such that $x>N\Rightarrow\sqrt{x}>M$. In the language of predicate logic, prove

$$(\forall M > 0)(\exists N > 0)(\forall x \in \mathbb{R})\left(x > N \Rightarrow \sqrt{x} > M\right).$$

**Ans:** For an arbitrary positive integer $M$ , we pick $N = M^2$ which yields $x > N = M^2 \Rightarrow \sqrt{x} > M$ . Congratulations, you just proved $\lim_{x\to\infty} \sqrt{x} = \infty$ .

8. **(Doing Mathematics)** Sometimes textbooks lead one to believe that statements of the form $P \Rightarrow Q$ or $P \Leftrightarrow Q$ appear magically out of thin air for mathematicians to prove. If this were so, which of course it isn't, mathematics would be a purely deductive science, but in fact "doing mathematics" and mathematical research is as much an inductive science as deductive. The following table below lists the number of divisors $\tau(n)$ of the first 24 natural numbers. (For example the divisors of 6 are 1, 2, 3, and 6 for a total of $\tau(6) = 4$ divisors). Looking at the table can you think of questions to ask about the number of divisors of the natural numbers? What would be some "candidate" theorems you would like to verify? Maybe you can think of a theorem and then prove it. A couple possible theorems that might be worth proving (or disproving) are

Theorem 1: $\tau(n)$ is odd if and only if $n$ is a square, like 4, 9, 16, 25, … .

Theorem 2: If $m$ and $n$ have no common factor, then $\tau(m)\tau(n) = \tau(mn)$.

| $n$ | $\tau(n)$ | | $n$ | $\tau(n)$ |
|---|---|---|---|---|
| 1 | 1 | | 13 | 2 |
| 2 | 2 | | 14 | 4 |
| 3 | 2 | | 15 | 4 |
| 4 | 3 | | 16 | 5 |
| 5 | 2 | | 17 | 2 |
| 6 | 4 | | 18 | 6 |
| 7 | 2 | | 19 | 2 |
| 8 | 4 | | 20 | 6 |
| 9 | 3 | | 21 | 4 |
| 10 | 4 | | 22 | 4 |
| 11 | 2 | | 23 | 2 |
| 12 | 6 | | 24 | 8 |

Divisors of a Few Natural Numbers

9. **(Number Game)** Show that if the number $d_1d_2d_3$ is a multiple of 9 if and only if $d_1 + d_2 + d_3$. Is a multiple of 9.

**Ans:**

If $d_1 d_2 d_3 = 9k$
hence $d_1 \times 10^2 + d_2 \times 10 + d_3 = 9k$
hence $d_1 (9+1)^2 + d_2 (9+1) + d_3 = 9k$
hence $d_1 (9^2 + 2\times 9 + 1) + d_2 (9+1) + d_3 = 9k$
hence $d_1 + d_2 + d_3 = 9k - d_1 \times 9^2 - d_1 \times 2 \times 9 - 9 \times d_2$
$= 9(k - 11d_1 - d_2)$
$= 9k_1$ where $k_1 \in \mathbb{N}$

Starting with $k_1 + k_2 + k_3 = 9k$ and working backwards, we have

If $d_1 + d_2 + d_3 = 9k$
hence $d_1 + d_2 + d_3 = 9k + (d_1 \times 9^2 - d_1 \times 9^2) + (18d_1 - 18d_1) + (9d_2 - 9d_2)$
hence $d_1 (9^2 + 18 + 1) + d_2 (9+1) + d_3 = 9(k + d_1 \times 9^2 + 18 d_1 + 9d_2)$
hence $d_1 \times 10^2 + d_2 \times 10 + d_3 = 9k_1$ where $k_1 \in \mathbb{N}$
hence $d_1 d_2 d_3 = 9k_1$

BHKPΞΩ