Problems 1.6 Proofs by Mathematical Induction

1. (Proof by Induction) Prove by weak or strong induction the following propositions.

a) $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

Ans: Here $P(1): 1^2=\dfrac{1\cdot(1+1)(2\cdot 1+1)}{6}$ is clearly true. We now assume

$P(n): 1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$ and add $(n+1)^2$ to each side of this equation, and performing some elementary algebra yielding the desired result $P(n+1)$:

$$
\begin{aligned}
P(n+1): 1^2+2^2+3^2+\cdots+n^2+(n+1)^2 &= \frac{n(n+1)(2n+1)}{6}+(n+1)^2 \\
&= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} \\
&= \frac{(n+1)\left[n(2n+1)+6(n+1)\right]}{6} \\
&= \frac{(n+1)(n^2+7n+6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)(n+2)\left[2(n+1)+1\right]}{6}
\end{aligned}
$$

b) $P(n): 1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$

Ans: Here $P(1): 1^3=\dfrac{1^2(1+1)^2}{4}$ is clearly true. We now assume

$P(n): 1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$ and add $(n+1)^3$ to each side of this equation, and performing some elementary algebra yielding the desired result $P(n+1)$:

$$
\begin{aligned}
P(n+1):\ 1^3+2^3+3^3+\cdots+n^3+(n+1)^3 &= \frac{n^2(n+1)^2}{4}+(n+1)^3 \\
&= \frac{n^2(n+1)^2+4(n+1)^2}{4} \\
&= \frac{(n+1)^2\left[n^2+4(n+1)\right]}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4}
\end{aligned}
$$

c) $1+3+5+\cdots+(2n-1)=n^2$.

Ans: It is clear that $P(1):1=1^1$. We now assume

$$P(n):1+3+5+\cdots+(2n-1)=n^2.$$

Adding $2n+1$ to each side of the equation gives the equation $1+3+5+\cdots+(2n-1)+(2n+1)=n^2+2n+1$. But this equation is simply

$$P(n+1):\ 1+3+5+\cdots+(2n-1)+\left[2(n+1)-1\right]=n^2$$

which proves the result.

d) 9^n-1 is divisible by 8 for all natural numbers n .

Ans: Clearly the result holds for $n=1$. Assuming the result holds for n means we can write $9^n-1=8k$ some integer k . We now must show that $9^{n+1}-1=8k_1$, where k_1 is some integer. We write

$$9^{n+1}-1=9(9^n-1)+8=9(8k)+8=8(9k+1)$$

which shows that 8 divides $9^{n+1}-1$, and by induction the desired result is proven.

e) For $n\geq 1,\ 1+2^2+2^3+2^4+\cdots+2^n=2^{n+1}-1$

Ans: Clearly the result holds when $n=1$ since we have $1=2^2-1$. We now assume the result $P(n)$: $1+2^2+2^3+2^4+\cdots+2^n=2^{n+1}-1$ and add 2^{n+1} to each side of the equation, getting the desired result:

$$P(n+1): 1+2^2+2^3+2^4+\cdots+2^n+2^{n+1} = \left(2^{n+1}-1\right)+2^{n+1}$$
$$= 2\cdot 2^{n+1}-1$$
$$= 2^{n+2}-1$$

f) For $n \ge 5,\ 4n < 2^n$,

Ans: We start the induction at $n=5$ which is clearly true since $4\cdot 5 = 20$ and $2^5 = 32$. We now assume the result $P(n)$: $4n < 2^n$ and add $4n$ to each side of the inequality, getting $4n+4 < 2^n+4$ or the desired result:

$$P(n+1): 4(n+1) < 2^n+4 < 2^n+2^n = 2\cdot 2^{n+1} = 2^{n+1}$$

g) $n^3 - n$ is divisible by 3 for $n \ge 1$.

Ans: The claim is clearly true for $n=1$ since 3 divides $3^3-3=24$. Using simple induction we assume 3 divides n^3-n is true for an arbitrary integer $n \ge 1$. Hence, there exists an integer k so that $n^3-n=3k$ and seek to prove $(n+1)^3-(n+1)=3k_1$ for some integer k_1. We do this by writing

$$(n+1)^3-(n+1) = \left(n^3+3n^2+3n+1\right)-(n+1)$$
$$= n^3+3n^2+2n$$
$$= \left(n^3-n\right)+\left(3n^2+3n\right)$$
$$= 3k+3n(n+1)$$
$$= 3\left[k+n(n+1)\right]$$
$$= 3k_1$$

Where $k_1 = k+n(n+1) \in \mathbb{N}$. Hence 3 divides $(n+1)^3-(n+1)$

h) $2^{n-1} \le n!,\ n \in \mathbb{N}$

Ans: When $n=1$ we have $2^{1-1}=2^0=1,\ 1!=1$ and so the condition holds. We now assume $P(n): 2^{n-1} \le n!$ and prove $P(n+1): 2^n \le (n+1)!$ To do this we write

$$(n+1)! = (n+1)n!$$
$$\ge (n+1)2^{n-1}$$
$$\ge 2\cdot 2^{n-1} \qquad (\text{since } n+1 \ge 2)$$
$$= 2^n$$

which proves the result.

i) For all positive integers n, n^2+n is even.

Ans: Clearly the assertion is true when $n=1$ since $1^2+1=2$**.** We now assume $P(n): n^2+n$ is even and prove $(n+1)^2+(n+1)$ is even . Since n^2+n is even we can write $n^2+n=2k$ for some integer k. Hence we can write

$$\begin{aligned}(n+1)^2+(n+1) &= n^2+2n+1+n+1 \\ &= (n^2+n)+(2n+2) \\ &= 2k+2(n+1) \\ &= 2(k+n+1) \\ &= 2k_1\end{aligned}$$

where $k_1 = k+n+1 \in \mathbb{Z}$. Hence by induction n^2+n is even for all natural numbers n.

j) For real numbers a,b and every natural number n: $(ab)^n = a^n b^n$

Ans: Clearly $P(1): (ab)^1 = a^1 b^1$ (true)

$P(n): (ab)^n = a^n b^n$ is true. Multiplying each side of this equation by ab gives

$(ab)^n ab = a^n b^n ab$. But the expression on the left is simply $(ab)^{n+1}$ and the expression on the right can be written as $a^n b^n ab = a^n a b^n b = a^{n+1} b^{n+1}$, which completes the proof.

2. **(Something Fishy)** Suppose we hypothesize that $P(n): n^2+7n+3$ is an even integer for $n=1,2,\ldots.$ where we seek to prove it by induction. Is there something wrong the following induction argument? We assume $P(n): n^2+7n+3$ to be even and so $n^2+7n+3=2k$ for some integer k. Hence we have

$$\begin{aligned}(n+1)^2+7(n+1)+3 &= (n^2+2n+1)+7n+7+3 \\ &= (n^2+2n+3)+2(n+4) \\ &= 2k+2(n+4) \\ &= 2(k+n+4) \\ &= 2k_2\end{aligned}$$

where $k_1 = k + n + 4 \in \mathbb{N}$ which proves the induction hypothesis. Hence $n^2 + 7n + 3$ is even for all natural numbers n.

Ans: The base step $P(1): 1^2 + 7(1) + 3 = 11$ is not an even number so induction fails. In fact you can prove that all the terms in the sequence $n^2 + 7n + 3$ are *odd,* and you don't need to use induction, simply observe $n^2 + 7n + 3 = n(n+7) + 3$, and so if n is odd we have

$$\begin{aligned}\text{odd}(\text{odd}+7)+3 &= \text{ odd }(\text{even})+\text{odd} \\ &= \text{even}+\text{odd} \\ &= \text{odd}\end{aligned}$$

By a similar argument if n is even we get $n^2 + 7n + 3$ to be odd.

3. **(Clever Mary)** To prove the identity

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Mary evaluated the left-hand side of the equation for $n = 0,1,2$ getting

n	0	1	2
$F(n)$	0	1	3

.

She then fit a polynomial to these three points, getting

$$f(n) = \frac{n(n+1)}{2}.$$

Mary turned this into her professor. Is her proof[1] valid?

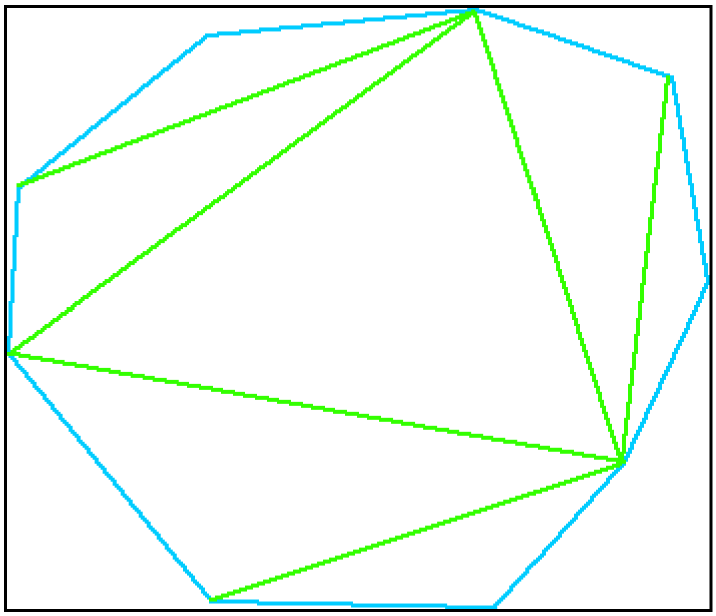
Ans: A rather unusual way to look at this identify, but it is easy to see that both sides of the identity satisfy the same recurrence relation $p(n) = p(n-1) + n,\ p(1) = 1$, where $p(n) = 1 + 2 + \cdots + n$. Hence, the identity holds for all natural numbers n.

4. **(Hmmmmmmmm)** Is there something fishy with this argument that Mary can carry a 50-ton load of straw on her back. Clearly she can carry one straw on her back, and if she can carry n straws on her back, she can certainly one more. Hence, she can carry any number of straws on her back which can amount to a 50-ton load.

[1] This problem is based on a problem in the book $A = B$ by Marko Petkovsek, Doron Zeilberger and Herbert Wilf. (This amazing book, incidentally, can be downloaded free on the internet.)

Ans: There is something to the old adage that it was a straw that broke the camel's back, but in this case it was Mary's back. If we allow this induction step to hold, we could prove all *sorts* of things. You can make up a few outrageous claims yourself.

5. **(Geometric Principle by Induction)** Show that every convex polygon[2] can be divided into triangles (i.e. triangulated). An example illustrating a triangulation (triangulations are not unique) of a 8-sided convex polygon is shown below.



Ans: The trivial case is the observation that every convex polygon with $n = 3$ sides is a triangle, and so it is triangulated as it stands. We now assume any convex polygon with n sides can be triangulated and prove that a convex polygon with $n+1$ sides can be triangulated. So starting with an arbitrary convex polygon with $n+1$ sides, draw a line connecting two vertices of adjacent sides, arriving at a triangle and a polygon with n sides. We then triangulate the polygon with n sides and combining this with the new triangle, we have triangulated the $(n+1)$-polygon. Hence, we have proven the induction hypothesis and hence any convex polygon with n sides can be triangulated.

6. **(Nature of Induction)** Often one gets a general idea something is true by constructing examples. The idea then is to prove your hypothesis by induction. For example, suppose you have only 3 and 5 cents stamps and want to determine what postages are possible. So you make a table like

[2] A convex polygon is a simple polygon (sides do not cross) whose interior is a convex set.(i.e the line segment connecting every pair of points in the set also belongs to the set.

		5 CENT STAMP					
		0	1	2	3	4	5
3 CENT STAMP	0	0	5	10	15	20	…
	1	3	8	13	18	23	…
	2	6	11	16	21	26	…
	3	9	14	19	24	29	…
	4	12	17	22	27	32	…
	5	…	…	…	…	…	…

From this table we might hypothesize that possible postages are 0,3,5, and 6 cents and every value of 8 or more cents. Can you prove this by induction?

Ans: If we denote

$P(n)$: postage of n cents is possible

then the goal is to prove $P(n)$ for $n=0,1,2,\ldots$. We can verify the initial step and show $P(n)$ is true for $n=0,3,5,$ and 6 and false for $n=1,2,4,5,$ and 7. We now resort to strong induction and *assume* all postages are possible for $8,9,\ldots,n$ cents and prove a postage worth $n+1$ cents is possible. To do this consider four cases:

case 1: $n+1=8$ (true since we can use one 3 cent stamp and one 5 cent stamp)
case 2: $n+1=9$ (true since we can use three 3 cent stamps)
case 3: $n+1=10$ (true since we can use two 5 cent stamps)
case 4: $n+1>10$ (this is the same as $n-2\geq 8$ which is assumed true by strong induction)

So we have proven $P(0)\wedge P(1)\wedge P(2)\wedge\cdots P(n)\Rightarrow P(n+1)$ and so by strong induction $P(n)$ is true for all $n=0,1,2,\ldots$ Note: The basic idea is that for $n>10$ (or $n=11,12,13,\ldots$) we know postage of $n-2=8,9,10,\ldots$ (three cents less) is possible, so we simply add another 3 cent stamp to get a postage of $n+1$.

7. **(Fibonacci Sequence)** The Fibonacci sequence $\{F_n, n=1,2,\ldots\}$ is defined for $n\geq 2$ by the equations $F_{n+1}=F_n+F_{n-1},\ F_1=1, F_2=1$. A few terms of the sequence are 1,1,2,3,5,8,13,… . Show that the nth term of the sequence is given by

$$F_n=\frac{\alpha^n-\beta^n}{\sqrt{5}}$$

where $\alpha=\left(1+\sqrt{2}\right)/2,\ \beta=\left(1-\sqrt{2}\right)/2$.

Ans: We first verify the result for $n = 1, 2$. When $n = 1$ the formula for F_n yields $F_1 = 1$ as desired. When $n = 2$ after some simple algebra the formula yields

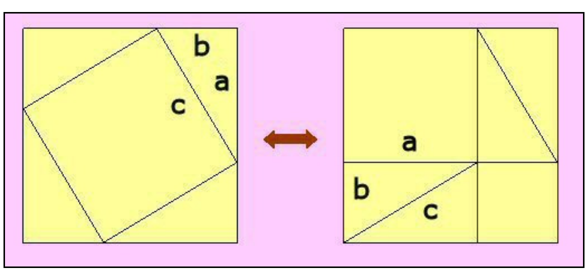
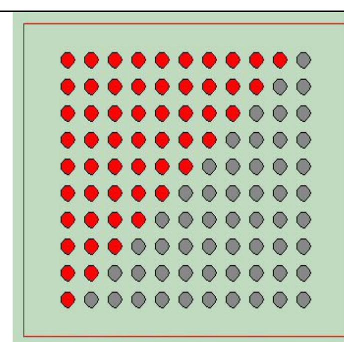
$$F_2 = \frac{\alpha^2 - \beta^2}{\sqrt{5}} = \frac{\alpha - \beta}{\sqrt{5}} = 1$$

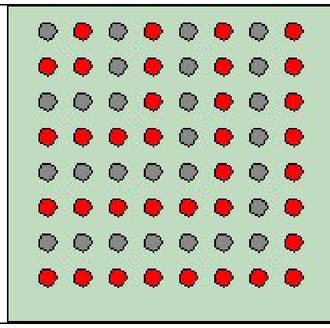
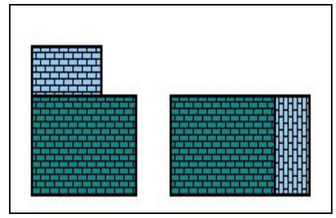
as desired, using the facts that $\alpha^2 = \alpha + 1, \beta^2 = \beta + 1$. We now use strong induction and assume the formula holds for all values $1, 2, 3, \ldots, n$ and we wish to show the equation holds for $n+1$. We do this by writing

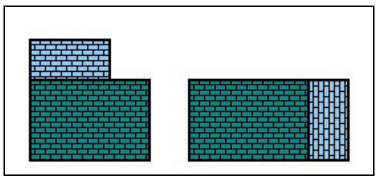
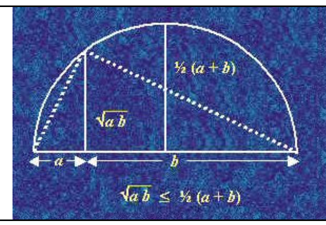
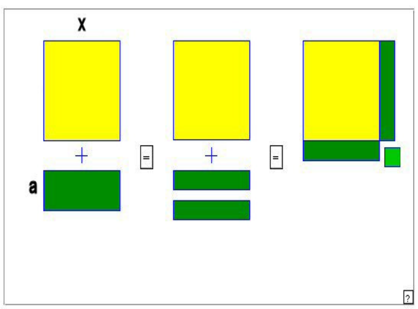
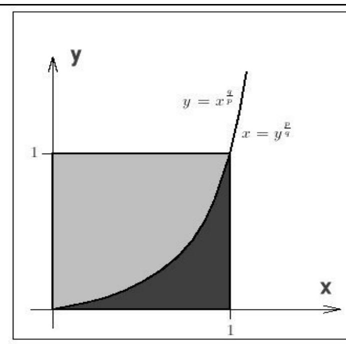
$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &= \frac{\alpha^n - \beta^n}{\sqrt{5}} + \frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}} \quad \text{(by induction)} \\ &= \frac{\left[\alpha^{n-1}(1+\alpha) - \beta^{n-1}(1+\beta)\right]}{\sqrt{5}} \\ &= \frac{\left[\alpha^{n-1}\alpha^2 - \beta^{n-1}\beta^2\right]}{\sqrt{5}} \quad (\text{using } \alpha^2 = \alpha + 1, \beta^2 = \beta + 1) \\ &= \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}} \end{aligned}$$

which proves the result for $n+1$. Hence, by strong induction the result holds for all natural numbers n.

8. **(Proofs without Words)** They say a good picture is worth a thousand words, but in mathematics it might be closer to a million. For the figures below, describe why the figure provides a visual proof of the statement.

	
a) $a^2 + b^2 = c^3$	b) $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

	
c) $1+3+5+\cdots+(2n-1)=n^2$	d) $x^2-y^2=(x-y)(x+y)$

	
e) $x^2-y^2=(x-y)(x+y)$	f) $\sqrt{ab}\le\dfrac{a+b}{2}$
	
g) $x^2+ax=\left(x+\dfrac{a}{2}\right)^2-\left(\dfrac{a}{2}\right)^2$	h) $\int_0^1\left(t^{p/q}+t^{q/p}\right)dt=1$

Ans: We leave it to the reader to have fun with these.

9. **(Peano's Axioms)** The Principle of Mathematical Induction is generally taken as an axiom for the natural numbers. It can be proven however from more basic axioms, namely **Peano's axioms**. Google Peano's axioms and read about them on the internet.

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