Problems 2.1: Basic Operations of Sets

1. **(Set Notation)** Write the following sets in set notation $\{x : P(x)\}$.

a) The real numbers between 0 and 1.
Ans: $\{x \in \mathbb{R} : 0 < x < 1\}$

b) The natural numbers between 2 and 5.
Ans: $\{3,4\}$

c) The set of prime numbers.
Ans: $\{n \in \mathbb{N} :$ only divisors of n are itself and $1\}$

d) $\{1,2,3,...\}$
Ans: $\{n : n \in \mathbb{N}\}$

e) $\{5,6,7\}$
Ans: $\{n \in \mathbb{N} : 5 \le n \le 7\}$

f) The solutions of the equation $x^2 - 1 = 0$.
Ans: $\{x \in \mathbb{R} : x^2 - 1 = 0\}$

2. Tell if the following statements are true or false if $A = \{\{a\},\{b,c\},\{d,e,f\}\}$.

a) $a \in A$ **Ans:** F
b) $a \subseteq A$ **Ans** F
c) $c \in A$ **Ans:** F
d) $\{b,c\} \in A$ **Ans:** T
e) $\varnothing \in A$ **Ans:** F
f) $\varnothing \subseteq A$ **Ans:** T

3. **(Checking Subsets: True or False?)**

a) $\mathbb{Z} \subseteq \mathbb{R}$ **Ans:** T
b) $\mathbb{R} \subseteq \mathbb{C}$ **Ans:** T
c) $(0,1) \subseteq [0,1]$ **Ans:** T
d) $(0,1) \subseteq \mathbb{R}$ **Ans:** T
e) $(2,5) \subseteq \mathbb{Q}$ **Ans:** F
f) $\mathbb{Q} \subseteq (2,5)$ **Ans:** F

g) $[1,3]\subseteq\{1,3\}$ **Ans:** F

h) $\{1,3\}\subseteq[1,3]$ **Ans:** T

i) $\{3,15\}\subseteq\{3,5,7,15\}$ **Ans:** T

4. **(The Empty Set: True or False?)**

a) $\varnothing=\{\varnothing\}$ **Ans:** F

b) $\varnothing\in\{\varnothing\}$ **Ans:** T

c) $\varnothing\subseteq\{\varnothing\}$ **Ans:** T

d) $A\cup\varnothing=A$ **Ans:** T

e) $\{\varnothing\}\subseteq\varnothing$ **Ans:** F

f) $\{\varnothing\}\in\{\{\varnothing\}\}$ **Ans:** T

g) $\{\{\varnothing\}\}\in\{\varnothing,\{\varnothing\}\}$ **Ans:** T

5. **True or False?**

a) $A\in A$ **Ans:** F

b) If $A\subseteq B$ and $x\notin B$ then $x\notin A$ **Ans:** T

c) If $A\subseteq B$ then $A\in B$. **Ans:** F

d) If $A\in B$ then $A\subseteq B$ **Ans:** F

e) If $A\in B$ and $B\in C$ then $A\in C$ **Ans:** F

f) If $A\in B$ and $B\in C$ then $A\subseteq C$ **Ans:** F

6. **(Sets, Members and Subsets)** Fill in the blank with one of the combinations $(\in,\subseteq),(\in,\not\subseteq),(\notin,\subseteq),(\notin,\not\subseteq)$ that describe the given relation.

a) a ___ $\{c,a,t\}$ **Ans**: $\in,\ \not\subseteq$

b) a ___ $\{c,a,\{a\},t\}$ **Ans:** $\in,\ \not\subseteq$

c) $\{a,t\}$ ___ $\{c,a,t\}$ **Ans**: $\notin,\ \subseteq$

d) $\{a\}$ ___ $\{c,\{a\},t\}$ **Ans**: $\in,\ \not\subseteq$

e) $\{a,\{t\}\}$ ___ $\{c,a,t,\{t\}\}$ **Ans**: $\notin,\ \subseteq$

f) $\{a,\{t\}\}$ ___ $\{c,a,t,\{t\},\{a,\{t\}\}\}$ **Ans**: $\in,\ \subseteq$

g) $\{c,\ a,\{t\}\}$ ___ $\{a,t,\{t\}\}$ **Ans**: $\notin,\ \not\subseteq$

h) $\{a,\{t\}\}$ ___ $\{c,a,t,\{t\}\}$ **Ans:** $\notin,\ \subseteq$

i) $\varnothing$ ___ $\varnothing$ **Ans**: $\notin,\ \subseteq$

j) $\{\varnothing\}$ ___ $\{\varnothing\}$ **Ans**: $\notin$, $\subseteq$

k) $\{\varnothing\}$ ___ $\{\varnothing,\{\varnothing\}\}$ **Ans**: $\in$, $\subseteq$

l) $\varnothing$ ___ $\{\varnothing\}$ **Ans**: $\in$, $\subseteq$

m) $\{\varnothing\}$ ___ $\varnothing$ **Ans:** $\notin$, $\not\subseteq$

7. **(Power Sets)** Find the power set of the given sets.

a) $A=\{4,5,6\}$

Ans: $P(A)=\{\varnothing,\{4\},\{5\},\{6\},\{4,5\},\{4.6\}\{5,6\},\{4,5,6\}\}$

b) $A=\{\oplus,\odot,\otimes\}$

Ans:

$P(A)=\{\varnothing,\{\oplus\},\{\odot\},\{\otimes\},\{\oplus,\odot\},\{\oplus,.\otimes\}\{\odot,\otimes\},\{\oplus,\odot,\otimes\}\}$

c) $A=\{a,\{b\}\}$

Ans: $P(A)=\{\varnothing,\{a\},\{\{b\}\},\{a,\{b\}\}\}$

d) $A=\{a,\{b,\{c\}\}\}$

Ans: $P(A)=\{\varnothing,\{a\},\{\{b,\{c\}\}\},\{a,\{b,\{c\}\}\}\}$

e) $A=\{a,\{a\}\}$

Ans: $P(A)=\{\varnothing,\{a\},\{\{a\}\},\{a,\{a\}\}\}$

f) $A=\{\varnothing,\{\varnothing\}\}$

Ans: $P(A)=\{\varnothing,\{\varnothing\},\{\{\varnothing\}\},\{\varnothing,\{\varnothing\}\}\}$

8. **(Matching Sets)** Which pairs of the following sets are connected by one or more of the

relations $\in,\subseteq,$ or $\supseteq$?

a) $\mathbb{R}$

Ans:

$3\in \mathbb{R}$, $\{1,2,3,\ldots,10\}\subseteq\mathbb{R},\{x : x \text{ is an even integer}\}\subseteq\mathbb{R},(-1,1)\subseteq\mathbb{R},\varnothing\subseteq\mathbb{R}$

b) 3

Ans: $3 \in \mathbb{R},\ 3 \in \{1,2,3,\ldots,10\}$

c) $\{1,2,3,...,10\}$
Ans: $\{1,2,3,\ldots,10\} \subseteq \mathbb{R}, 3 \in \{1,2,3,\ldots,10\}$,

d) $\{x : x \text{ is an even integer}\}$
Ans: $\{x : x \text{ is an even integer}\} \subseteq \mathbb{R}, \varnothing \subseteq \{x : x \text{ is an even integer}\}$

e) $(-1,1)$
Ans: $(-1,1) \subseteq \mathbb{R}$

f) $\varnothing$
Ans: $\varnothing$ is a subset of all the other sets, 3 is not related to $\varnothing$

9. **(Find the Set)** Let $A = \{a_1, a_2, a_3, ...\}$ were a_n is the remainder of n divided by 5. List the
elements of A.
Ans: $A = \{0,1,,2,3,4\}$

10. **(Interesting)** If $A = \{a,b,c\}$ is $A \in P(A)$? Is $A \subseteq P(A)$?
Ans: Here $P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\{b,c\}, \{a,b,c\}\}$ and so $A \notin P(A)$ but $A \subseteq P(A)$.

11. **(Power Set as a Collection of Functions)** The power set of a set can be interpreted as the set of all functions[1] defined on the set whose values are 0 and 1. For example, the functions defined on $A = \{a,b\}$ with values 0 and 1 are

- $f(a) = 0, f(b) = 0$ corresponds to $\varnothing$
- $f(a) = 0, f(b) = 1$ corresponds to $\{b\}$
- $f(a) = 1, f(b) = 0$ corresponds to $\{a\}$
- $f(a) = 1, f(b) = 1$ corresponds to $\{a,b\}$

Show that the elements of the power set of $A = \{a,b,c\}$ can be placed in this one-to-one correspondence with the functions on A whose values are either 0 or 1.

Ans: The power set of A is

[1] Although we haven't introduced functions yet in this book, we are sure most readers have familiarity with the subject.

$$P(A)=\{\varnothing,\{a\},\{b\},\{c\},\{a,b\},\{a,c\}\{b,c\},\{a,b,c\}\}$$

A 1-1 correspondence is

$f(a)=0, f(b)=0, f(c)=0$ corresponds to $\varnothing$
$f(a)=0, f(b)=0, f(c)=1$ corresponds to $\{c\}$
$f(a)=0, f(b)=1, f(c)=0$ corresponds to $\{b\}$
$f(a)=0, f(b)=1, f(c)=1$ corresponds to $\{b,c\}$
$f(a)=1, f(b)=0, f(c)=0$ corresponds to $\{a\}$
$f(a)=1, f(b)=0, f(c)=1$ corresponds to $\{a,c\}$
$f(a)=1, f(b)=1, f(c)=0$ corresponds to $\{a,b\}$
$f(a)=1, f(b)=1, f(c)=1$ corresponds to $\{a,b,c\}$

12. **(Second Power Set)** The second power set of a set A is the set of subsets of the set of subsets of the set, or $P(P(A))$. What is the second power set of $A=\{a\}$?
Ans: The first power set is $P(A)=\{\varnothing,\{a\}\}$ and so second power set is

$$P(P(A))=\{\varnothing,\{\varnothing\},\{\{a\}\},\{\varnothing,\{a\}\}\}.$$

13. **(Power Set of the Empty Set)** Prove $P(\varnothing)=\{\varnothing\}$
Ans: Since $\varnothing\subseteq\varnothing$ and so $\varnothing\notin P(\varnothing)$ and therefore $\{\varnothing\}\subseteq P(\varnothing)$. Conversely, if $A\in P(\varnothing)$ then $A\subseteq\varnothing$ which implies $A=\varnothing$ or $A\in P(\varnothing)$. Hence we have $P(\varnothing)=\{\varnothing\}$'

14. **(The Man Who Constructed Something from Nothing)** The German mathematician Leopold Kronecker once said, *"God created the integers and all else is the work of man."* But logicians like to say that logicians created the integers from more basic axioms. The German logician Gottlieb Frege defined the integers from nothing; i.e. using only the *empty set!* How did he do it? He defined recursively:

$$\boxed{0=\varnothing,\ 1=\{0\},\ 2=\{1\},\ 3=\{2\},\ \cdots}$$

and so on. That is, 0 is empty, 1 is the singleton $\{0\}$, 2 is the pair $\{0,1\}$, and so on. The idea for this construction is that the number of elements in the sets 0,1,2, .. is, respectively, zero, one, two, … Any thoughts on how you would would construct an "arithmetic" using this definition. How would you define "1 + 1" so you would get 2? What about "1 + 3"?

Ans: Each natural number is defined to be the set whose only member is the previous number, such as $3=\{2\}$. So $n+1=\{n\}$ for any natural number n. In other words $1+1=\{1\}=2$, $3+1=\{3\}=4$, $3+2=3+(1+1)=(3+1)+1=4+1=5$. .

15. Let A, B and C be arbitrary subsets of a universe U. Prove the following.

a) $A \subseteq A$
Ans: Let $a \in A$, hence $a \in A$. Hence $A \subseteq A$

b) $A \cup \varnothing = A$
Ans: $(\subseteq)$ Let $a \in A \cup \varnothing$. Therefore $a \in A$ or $a \in \varnothing$. **But** $a \notin \varnothing$ **so** $a \in A$. **Hence** $A \cup \varnothing \subseteq A$.

$(\supseteq)$ Let $a \in A$. Therefore a belongs to A or $\varnothing$. Therefore $a \in A \cup \varnothing$. **Hence** $A \subseteq A \cup \varnothing$. Hence, we have $A \cup \varnothing = A$.

c) $A \cap \varnothing = \varnothing$
Ans: $(\subseteq)$ Let $a \in A \cap \varnothing$. Therefore $a \in A$ and $a \in \varnothing$. But there is no element that belong to $\varnothing$ and so no element that belongs to both and so $A \cap \varnothing \subseteq \varnothing$.

$(\supseteq)$ Let $\varnothing \subseteq A \cap \varnothing$ since the empty set is a subset of any set. Hence, we have equality $A \cap \varnothing = A$.

d) $\varnothing = \overline{U}$
Ans: The universe U is defined as all elements under consideration, so by definition there is nothing outside U. Thus $\varnothing = \overline{U}$.

e) $A \cap U = A$
Ans: $(\subseteq)$ Let $a \in A \cap U$. Hence $a \in A$ and $a \in U$. Hence $a \in A$.

$(\supseteq)$ Let $a \in A$. But $A \subseteq U$ and so $a \in A \cap U$. Hence, we have $A \cap U = A$

g) $A \cap \overline{A} = \varnothing$
Ans: $(\subseteq)$ Let $a \in A \cap \overline{A}$. Hence $a \in A$ and $a \notin A$ which can not hold for any $a \in U$. Hence $A \cap \overline{A}$ is the empty set.

$(\supseteq)$ The empty set is a subset of any set and so $\varnothing \subseteq A \cap \overline{A}$. Hence, we have $A \cap \overline{A} = \varnothing$

h) $A \subseteq B \Rightarrow A \cup B = B$
Ans: We show $A \cup B = B$ using the condition $A \subseteq B$.

$(\subseteq)$ Let $x \in A \cup B$. Hence $x \in A$ or $x \in B$. But $A \subseteq B$ and so $x \in B$. Hence $A \cup B = B$

$(\supseteq)$ Let $x \in B$. Hence $x \in A$ or $x \in B$. Hence $x \in A \cup B$. Hence $A \cup B = B$.

i) $A \cup A = A \cap A$

Ans: $(\subseteq)$ Let $a \in A \cup A$. Hence $a \in A$ or $a \in A$. Hence $a \in A$ and $a \in A$. Hence $a \in A \cap A$.

$(\supseteq)$ This is true since if a belong to both A and A it belongs to A or A. Hence $A \cup A \supseteq A \cap A$. Hence $A \cup A = A \cap A$.

j) $\overline{\overline{A}} = A$

Ans: $(\subseteq)$ Let $a \in \overline{\overline{A}}$. Hence $a \notin \overline{A}$. Hence $a \in A$. Hence $\overline{\overline{A}} \subseteq A$.

$(\supseteq)$ We have $a \in A \Rightarrow a \notin \overline{A} \Rightarrow a \in A$. Hence $\overline{\overline{A}} \subseteq A$. Hence, we have $\overline{\overline{A}} = A$.

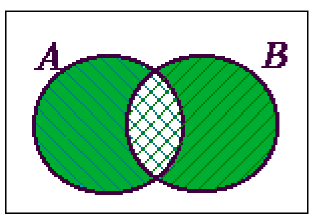
16. **(Difference Between Sets)** The formula $A - B = A \cap \overline{B}$ defines the difference between two sets in terms of the operations of intersection and complement. Can you find a formula for the union $A \cup B$ in terms of intersections and complements?

Ans: $A \cup B = \overline{\left(\overline{A} \cap \overline{B}\right)}$

17. **(Symmetric Difference)** The **symmetric difference** to two sets A and B is the set of elements that belong to one of the sets but not both and is denoted by

$$A \Delta B = (A - B) \cup (B - A).$$

It is both commutative and associative and is a kin to the exclusive OR in logic. See the following figure.



$A \Delta B$

Find the following symmetric differences where

$$E = \{2, 4, 6, \ldots\} \qquad \text{(even integers)}$$
$$O = \{1, 3, 5, \ldots\} \qquad \text{(odd integers)}.$$

a) $E \Delta O$

Ans: $E \Delta O = (E - O) \cup (O - E) = E \cup O = \mathbb{N}$

b) $E \,\Delta\, \varnothing$

Ans: $E \,\Delta\, \varnothing = (E-\varnothing)\cup(\varnothing-E) = E\cup\varnothing = E$

c) $E \,\Delta\, E$

Ans: $E \,\Delta\, E = (E-E)\cup(E-E) = \varnothing\cup\varnothing = \varnothing$

18. **(NASC for Disjoint Sets)** Prove $A\cap B=\varnothing \Leftrightarrow A-B=A$.
Ans: $(\Rightarrow)$ Assume $A\cap B=\varnothing$. First observe $A-B\subseteq A$. To $A\subseteq A-B$, let $a\in A$. But we are assuming $A\cap B=\varnothing$ and so $a\notin B$. Hence $A\subseteq A-B$.

$(\Leftarrow)$ If $A-B=A\cap\overline{B}=A$, then every member of A is in A but not in B. Hence $A\cap B=\varnothing$.

19. **(Distributive Law)** Prove that if A, B and C are sets, then "$\cup$" distributes over "$\cap$". That is

$$A\cup(B\cap C)=(A\cup B)\cap(A\cup C).$$

Ans: We write

$$\begin{aligned} A\cup(B\cap C) &= \{x : x\in A \text{ or } x\in B\cap C\} \\ &= \{x : x\in A \text{ or } (x\in B \text{ and } x\in C)\} \end{aligned}$$

We now use the analogous logical formula $p\vee(q\wedge r)=(p\vee q)\wedge(p\vee r)$ which states that "or" distributes over "and" to get the desired result.

20. **(Set Identity)** Prove $A\subseteq B \Leftrightarrow A\cup B=B$.
Ans: $(\Rightarrow)$ By definition $B\subseteq A\cup B$ since $x\in B\Rightarrow x\in A$ or $x\cup B$. Conversely, given that $A\subseteq B$, $x\in A\Rightarrow x\in B$ so that $x\in A\Rightarrow x\in A$ and $x\in B$. In other words $A\subseteq B$.

$(\Leftarrow)$ If $x\in A$ then $x\in A\cup B$. But we are assuming $A\bigcup B=B$ and so $x\cup B$ and so $A\subseteq B$.

21. **(Proving Set Relations with Truth Tables)** It is possible to identities of sets using truth tables. For example, one of DeMorgan's laws $\overline{(A\cup B)}=\overline{A}\cap\overline{B}$ can be verified with the following truth table, replacing the union of sets by the logical "or", set intersection by the logical "and" and set complementation by the logical "not," we see the membership values in columns (2) and (5) are the same.

		(1)	(2)	(3)	(4)	(5)
$x \in A$	$x \in B$	$A \cup B$	$\overline{(A \cup B)}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
T	T	T	**F**	F	F	**F**
T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

↑ ↑ same truth values

Prove the following identities using truth tables.

a) $A \cap \overline{A} = \varnothing$
b) $A \cup \overline{A} = U$
c) $A \cup (B \cup C) = (A \cup B) \cup C$ (associativity)
d) $A \cup B = (A \cap B) \cup (A - B) \cup (B - A)$

Ans: Direct result of true tables. The universal set is analogous to a tautology (i.e. all *T* values) and the empty set is equivalent to a contradiction (i.e. all *F* values).

22. **(Sets and Their Power Sets)** Prove

$$A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$$

Ans: $(\Rightarrow)$ Assume $A \subseteq B$ and let $X \in P(A)$. But $X \in P(A)$ means $X \subseteq A$ and by assumption $X \subseteq A \subseteq B$. But this means $X \in P(B)$ and we have proven $P(A) \subseteq P(B)$.

$(\Leftarrow)$ We now assume $P(A) \subseteq P(B)$ and follow the implications

$$x \in A \Rightarrow \{x\} \subseteq P(A) \Rightarrow \{x\} \in P(B) \Rightarrow \{x\} \subseteq B.$$

Hence $A \subseteq B$. Thus we have $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$.

23. **(Relationship Between Logic and Set Operations)** Using the fact that an implication in logic is equivalent to its contrapositive, show that for subsets A, B of a universal set *U*, we have $A \subseteq B \Leftrightarrow \overline{B} \subseteq \overline{A}$.
Ans: Use the idea discussed in the problem.

24. **(Computer Representation of Sets)** Finite sets can be represented efficiently by vectors of 0s and 1. For example, suppose we agree on an ordering of a set $U = \{a_1, a_2, \ldots, a_n\}$. We can represent a subset S of this set by a bit string where the ith bit is 1 if $a_i \in S$ and 0 if $a_i \notin S$. The following problems relate to this representation of sets. Take as the universe the set $U = \{0,1,2,3,4,5,6,7\}$.

a) For universe $U = \{3,9,2,5,6\}$, what is $S \subseteq U$ for the bit string 11001 ?

Ans: $S = \{2,9,6\} \subseteq U$

b) What is the bit string for $A = \{2,6\}$ if $U = \{1,2,3,4,5,6\}$?

Ans: bit string $= 010001$

c) What would be the representation of the intersection $S \cap T$ of two subsets of some universe?

Ans: Multiply binary strings term by terms using multiplication: $0 \cdot 0 = 0,\ 0 \cdot 1 = 1 \cdot 0 = 0,\ 1 \cdot 1 = 0$. (Or equivalently, take the minimum of each bit, term by term.)

d) What would be the representation of the intersection $S \cup T$ of two subsets of some universe ?

Ans: Add the binary strings term by term using addition: $0 + 0 = 0,\ 0 + 1 = 1 + 0 = 1,\ 1 + 1 = 1$

e) What would be the representation of the complement of a subset of some universe?

Ans: Interchange 0s and 1s in the bit representation.

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