Problems 2.2 Families of Sets

1. Let $A_1 = \{1,2\}, A_2 = \{2,3\}, A_3 = \{3,4\}$ and in general $A_k = \{k, k+1\}$. Write explicitly the
following sets.

a) $\bigcup_{k=1}^{5} A_k$ **Ans:** $\bigcup_{k=1}^{5} A_k = \{1,2,3,4,56\}$

b) $\bigcup_{k\in\mathbb{N}} A_k$ **Ans:** $\bigcup_{k\in\mathbb{N}} A_k = \mathbb{N}$

c) $\bigcup_{k\geq 5} A_k$ **Ans:** $\bigcup_{k\geq 5} A_k = \{n\in\mathbb{N} : n \geq 5\}$

d) $\bigcup_{1\leq k\leq 4} A_k$ **Ans:** $\bigcup_{1\leq k\leq 4} A_k = \{1,2,3,4,5\}$

e) $\bigcap_{k=1}^{5} A_k$ **Ans:** $\bigcap_{k=1}^{5} A_k = \varnothing$

f) $\bigcap_{k\in\mathbb{N}} A_k$ **Ans:** $\bigcap_{k\in\mathbb{N}} A_k = \varnothing$

2. Find the indexed families $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$ for the following sets.

a) $A_k = \left[0, \frac{k-1}{k}\right]$

Ans: $\bigcup_{k=1}^{\infty} A_k = [0,1)$, $\bigcap_{k=1}^{\infty} A_k = \{0\}$

b) $A_k = \left[-\frac{1}{k}, \frac{1}{k}\right]$

Ans: $\bigcup_{k=1}^{\infty} A_k = \mathbb{R}$, $\bigcap_{k=1}^{\infty} A_k = [-1,1]$

c) $A_k = \left(0, \frac{1}{k}\right)$

Ans: $\bigcup_{k=1}^{\infty} A_k = (0,1)$, $\bigcap_{k=1}^{\infty} A_k = \{0\}$

d) $A_k = \{k\} \cup \left[\frac{1}{k}, 2k\right]$

Ans: $\bigcup_{k=1}^{\infty} A_k = (0,\infty)$, $\bigcap_{k=1}^{\infty} A_k = [1,2]$

e) $A_k = [k, k+1]$

Ans: $\bigcup_{k=1}^{\infty} A_k = [1,\infty)$, $\bigcap_{k=1}^{\infty} A_k = \varnothing$

f) $A_k = \left[0, 1+\frac{1}{k}\right]$

Ans: $\bigcup_{k=1}^{\infty} A_k = [0,2]$, $\bigcap_{k=1}^{\infty} A_k = [0,1]$

g) $A_k = \{(x,y) : x^2 + y^2 \le k\}$

Ans: $\bigcup_{k=1}^{\infty} A_k = \mathbb{R}^2$, $\bigcap_{k=1}^{\infty} A_k = \{(x,y) : x^2 + y^2 \le 1\}$

3. **(Families of Sets in the Plan**e**)** Define a family of sets in the plane $\mathbb{R}^2$ by $A_{m,n} = \{(x,y) \in \mathbb{R}^2 : x \ge m, y \ge n\}$ where $a,b \in \mathbb{R}$. Find the following sets. Hint: Proceed exactly like one does with double series.

a) $\bigcup_{n=1}^{3}\left(\bigcup_{m=2}^{3} A_{a,b}\right)$

Ans:

$$\begin{aligned}\bigcup_{n=1}^{3}\left(\bigcup_{m=2}^{3} A_{a,b}\right) &= (A_{1,2} \cup A_{1,3}) \cup (A_{2,2} \cup A_{2,3}) \cup (A_{3,2} \cup A_{3,3}) \\ &= A_{1,2} \cup A_{1,3} \cup A_{2,2} \cup A_{2,3} \cup A_{3,2} \cup A_{3,3} \\ &= A_{1,2} \\ &= \{(x,y) \in \mathbb{R}^2 : x \ge 1, y \ge 2\}\end{aligned}.$$

b) $\bigcup_{n=1}^{3}\left(\bigcap_{m=2}^{3} A_{a,b}\right)$

Ans:

$$\bigcup_{n=1}^{3}\left(\bigcap_{m=2}^{3} A_{a,b}\right) = (A_{1,2} \cap A_{1,3}) \cup (A_{2,2} \cap A_{2,3}) \cup (A_{3,2} \cap A_{3,3})$$
$$= A_{1,2} \cup A_{2,3} \cup A_{3,3}$$
$$= A_{3,3}$$
$$= \{(x, y) \in \mathbb{R}^2 : x \geq 3, y \geq 3\}$$

4. **(Identity for an Indexed Family)** Show

$$B \cap \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right) = \bigcup_{\alpha\in\Lambda}(B \cap A_\alpha)$$

Ans: $(\subseteq)$ If $x \in B \cap \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)$ then $x \in B$ and $x \in \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)$. Since $x \in \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)$ there exists a $\beta \in \Lambda$ such that $x \in A_\beta$. It then follows that $x \in \bigcup_{\alpha\in\Lambda}(B \cap A_\alpha)$ .which proves

$$B \cap \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right) \subseteq \bigcup_{\alpha\in\Lambda}(B \cap A_\alpha).$$

$(\supseteq)$ To see $\bigcup_{\alpha\in\Lambda}(B \cap A_\alpha) \subseteq B \cap \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)$ let $x \in \bigcup_{\alpha\in\Lambda}(B \cap A_\alpha)$. Thus, there exists a $\beta \in \Lambda$ such that $x \in B \cap A_\beta$. Therefore $x \in B$ and $x \in A_\beta$. Since $x \in A_\beta$ it follows that $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$, and so $x \cup B \cap \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)$ and so it follows that $\bigcup_{\alpha\in\Lambda}(B \cap A_\alpha) \subseteq B \cap \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)$

5. **(Algebra of Sets)** Let A be a set. A family $\Im$ of subsets of A is called an **algebra**[1] of sets if

(i) $A \cup B$ is in Ψ whenever A and B are in Ψ

(ii) $\bar{A}$ is in Ψ whenever A is in Ψ

[1] Algebras of sets and sigma algebras (families of sets closed under *countable* unions) are fundamental to the study of measure theory. Note the difference between an algebra of subsets and a topology of subsets on a universe; just a minor difference makes for vastly different structures on the universe.

If this happens we say the family $\Im$ is **closed** under **unions** and **complementation**. Which of the following families of subsets of $A=\{a,b,c\}$ are an algebra of subsets of **A.**

a) The power set $\Im = P(A)$

Ans: Here the family is

$$P(A)=\{\varnothing,\{a\},\{b\},\{c\},\{a,b\},\{a,c\}\{b,c\},\{a,b,c\}\}$$

Note that the union and complement of any two of these sets in this family is again member of $P(A)$. Hence the family of subsets $P(A)$ is an algebra of subsets of A .

b) $\Im=\{\varnothing, A\}$

Ans: The union of any two of these members of this family: $\varnothing \cup A = A \in \Im$, $\varnothing \cap A = A \in \Im$ .Also $\overline{\varnothing} = A, \overline{A} = \varnothing$ and so tdhe family is closed under complementation. Hence the family of subsets $\{\varnothing, A\}$ of A is an algebra of subsets of A .

c) $\Im=\{\varnothing,\{a\}, A\}$

Ans: The family $\Im$ of subsets of A is not an algebra of sets since it is not closed under complementation since $\overline{\{a\}}=\{b.c\}\notin \Im$.

d) $\Im=\{\varnothing,\{a\},\{b,c\}, A\}$

Ans: This collection of subsets of A is an algebra since it is closed under unions as seen by

$\varnothing \bigcup \{a\}=\{a\}\in \Im$	$\overline{\varnothing} = A \in \Im$
$\varnothing \bigcup \{b,c\}=\{b,c\}\in \Im$	$\overline{\{a\}}=\{b,c\}\in \Im$
$\varnothing \bigcup A = A \in \Im$	$\overline{\{b,c\}}=\{a\}\in \Im$
$\{a\}\bigcup \{b,c\}=\{a,b,c\}=A\in \Im$	$\overline{A}=\varnothing \in \Im$
$\{a\}\bigcup A = A \in \Im$	
$\{b,c\}\cup A = A \in \Im$	

6. **(Sets of Length Zero)** In measure theory a subset A of the real numbers is said to have **length** (or **measure**) zero if $\forall \varepsilon > 0$ there exists a sequence $A_k=(a_k, b_k)$ of intervals such that they "cover" A : i.e.

$$A \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k)$$

and their *total length* is less than ε; that is

$$\sum_{k=1}^{\infty} |b_k - a_k| < \varepsilon \ .$$

Show that any sequence of real numbers $A = \{c_k\}$ has measure zero. Hint: Cover each element c_k in the sequence by an interval (a_k, b_k) of length $|b_k - a_k| = \varepsilon / 2^k$.

Ans: Let $\varepsilon > 0$ be an arbitrary positive real number and $\{r_k :, k = 1, 2, \ldots\}$ denote the rational numbers and (a_k, b_k) an interval containing r_k, where we assume the length of the kth interval is $b_k - a_k = \varepsilon / 2^k$. Hence, the total length of the covering intervals is

$$\text{total length of the covering } = \sum_{k=1}^{\infty} (b_k - a_k) = \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = \varepsilon$$

Hence the rational numbers have "length" zero.

7. **(Compact Sets)** A subset A of the real numbers is said to be **compact** if for every collection $\Im = \{(a_\alpha, b_\alpha) : \alpha \in \Lambda\}$ of open intervals that contains (or covers) A; i.e.

$$A \subseteq \bigcup_{\alpha \in \Lambda} (a_\alpha, b_\alpha)$$

there exists a *finite* subcollection of intervals of $\Im$ whose union also contains (covers) A. Show the set $A = (0,1)$ is not compact by showing the following.

a) A is covered by

$$\Im = \left\{ \left(0, 1 - \frac{1}{k} \right) : k = 1, 2, \ldots \right\}$$

b) There does not exist a finite subcollection of $\Im$ whose union contains A.

Ans: a) Since

$$(0,1) = \bigcup_{k=1}^{\infty} \left(0, \frac{k-1}{k} \right)$$

we have $(0,1)$ is covered by $\Im$.

b) However, if we take any finite collection of sets in $\Im$, there will be one set of the form

$$\left(0, \frac{K-1}{K} \right)$$

that contains the others and so the set $(0,1)$ is not contained in any finite union of such sets.

8. **(Topologies)** Verify that the following families are topologies for $\{a,b,c\}$.

$J_1=\{\varnothing,U\}$	indiscrete topology
$J_2=\{\varnothing,\{a\},U\}$	
$J_3=\{\varnothing,\{a\},\{b,c\},U\}$	
$J_4=\{\varnothing,\{a,b\},U\}$	
$J_5=\{\varnothing,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},U\}$	discrete topology

9. **(Topologies)** Which of the following families of subsets of $\{a,b,c\}$ are topologies on $\{a,b,c\}$?

a) $J=\{\varnothing,\{b\},\{c\},\{a,b,c\}\}$

Ans: J is not a topology for $\{a,b,c\}$ since the union $\{b\}\cup\{c\}=\{b,c\}\notin J$.

b) $J=\{\varnothing,\{a,b\},\{b,c\},\{a,b,c\}\}$

Ans: $J=\{\varnothing,\{a,b\},\{b,c\},\{a,b,c\}\}$ is not a topology for $\{a,b,c\}$ since the intersection $\{a,b\}\cap\{b,c\}=\{b\}\notin J$.

c) $J=\{\varnothing,\{a\},\{b,c\}\}$

Ans: $J=\{\varnothing,\{a\},\{b,c\}\}$ is not a topology for $\{a,b,c\}$ since the union $\{a\}\cup\{b,c\}=\{a,b,c\}\notin J$.

d) $J=\{\varnothing,\{b\},\{b,c\},\{a,b,c\}\}$

Ans: If all possible unions and intersections of sets in J are taken, the results will be sets in J. Hence, J is a topology for $\{a,b,c\}$.

10. **(Finding Intersections)** Find an example of an infinite family of sets, none of which is the given set, whose intersection is

a) $\{1\}$

b) $[0,\infty)$

Ans: a) $\{1\}=\bigcap_{k=1}^{\infty}\left[1-\frac{1}{k},1+\frac{1}{k}\right]=[0,2]\cap\left[\frac{1}{2},\frac{3}{2}\right]\cap\left[\frac{2}{3},\frac{4}{3}\right]\cap\cdots$

b) $[0,\infty)=\bigcap_{k=1}^{\infty}\left[-\frac{1}{k},\infty\right)=[-1,\infty)\cap\left[-\frac{1}{2},\infty\right)\cap\left[-\frac{1}{3},\infty\right)\cap\cdots$

11. **(Finding Unions)** Find an example of an infinite family of sets, none of which is the given set, whose union is

a) $(0,\infty)$

b) $\mathbb{R}$

Ans: a) $(0,\infty)=\bigcup_{k=1}^{\infty}(0,k)$

b) $\mathbb{R}=\bigcup_{k=1}^{\infty}(-k,k)$

12 .(**Quotient Set** $\mathbb{Z}/5$) Given the integers $\mathbb{Z}=\{0,\pm1,\pm2,\ldots\}$ we say that two integers are **equivalent** if they have the same remainder when divided by 5. Recalling that a negative number like -4 has a remainder of 1 when divided by 5 since

$$\frac{-4}{5}=\frac{-5+1}{5}=-1+\frac{1}{5} \text{ ‘}$$

find the family of sets where each set in the family consists of sets of equivalent integers.

This family of equivalent sets is an example of a **quotient set**, which in this case we denote by $\mathbb{Z}/5$.

Ans The quotient set is

$$\mathbb{Z}/5=\{[0],[1],[2],[3],[4]\}$$

where

$$[0]=\{0,\ \pm5,\ \pm10,\ \pm15,\ \ldots\}$$
$$[1]=\{1,\ -4,\ 6,-9,\ 11,\ \ldots\}$$
$$[2]=\{2,-3,\ 7,-8,\ 12,\ \ldots\}$$
$$[3]=\{3,\ -2,\ 8,\ -7,\ 13,\ -12,\ \ldots\}$$
$$[4]=\{4,\ -1,\ 9,\ -6,\ 14,\ -11,\ \ldots\}$$

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