Section 2.3 Counting: The Art of Enumeration

1. Compute the following.

a) $P(5,3)$ **Ans:** $5 \cdot 4 \cdot 3 = 60$

b) $P(4,1)$ **Ans:** 4

c) $P(30,2)$ **Ans:** $30 \cdot 29 = 870$

d) $C(4,1)$ **Ans:** $\frac{4!}{1!3!} = 4$

e) $C(10,8)$ **Ans:** $\frac{10!}{8!2!} = 45$

f) $\binom{7}{2}$ **Ans:** $\frac{7!}{2!5!} = 21$

g) $\binom{9}{2}$ **Ans:** $\frac{9!}{2!7!} = 36$

h) $(a+b)^6$ **Ans:**

$$(a+b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{0}b^6$$
$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

2. **(Distinguishable Permutations)** The number of **distinguishable permuations** of the word

TOOTS is $\frac{5!}{2!2!}$.

Since there are five letters in the word one writes $5!$ in the numerator. However, we cannot distinguish the 2 Ts and the 2 Os in the word, hence we divide 2!2!. Find the number of distinguishable permutations in the following words.

a) TO **Ans:** $\frac{2!}{1!1!} = 2$

b) TWO **Ans:** $\frac{3!}{1!1!} = 3$

c) TOO **Ans:** $\frac{3!}{2!1!} = 3$

d) BOOT **Ans:** $\frac{4!}{2!2!} = 6$

e) SNOOT **Ans:** $\frac{5!}{1!1!2!1!} = 60$

f) DALLAS **Ans:** $\frac{6!}{1!2!2!1!} = 120$

g) TENNESSEE **Ans:** $\frac{9!}{1!4!2!2!} = 3780$

h) ILLINOIS **Ans:** $\frac{8!}{3!2!1!1!} = 3360$

3. **(Going to the Movies)** Find the numbers of ways in which 4 boys and 4 girls can be seated in a row of 8 seats if they sit alternately and if there is a boy named Jerry and a girl named Susan who cannot sit next to each other.

Ans: Calling B = boy, G = girl we first count the arrangements of alternating boy and girl. We can arrange as

$$\text{B,G,B,G,B,G,B,G} = 4! \times 4! = 576 \text{ ways}$$
$$\text{G,B,G,B,G,B,G,B} = 4! \times 4! = 576 \text{ ways}$$

for a total of 576 + 576 = 1152 ways. We now must subtract from this number the number of ways where Jerry and Susan are sitting next to each other. The number of ways will be

$$\text{Jerry, Susan, B,G,B,G,B,G} = 6 \times 3! \times 3! = 252 \text{ ways}$$
$$\text{Susan, Jerry, G,B,G,B,G,B} = 6 \times 3! \times 3! = 252 \text{ ways}$$

For a total of 252 + 252 = 504 ways. To satisfy the condition that Jerry and Susan do not sit next to each other we must subtract 504. Hence the total number of desired arrangements is 1152 – 504 = 648 ways.

4. **(Movies Again)** Four couples go to the movies and sit together in six seats. How many ways can these people arrange themselves where each couple sits next to each other?

Ans: Four couples can arrangement themselves 4! Number of ways, where each couple can sit two ways. Hence, the total number of ways is $4! \times 2 \times 2 \times 2 = 192$ ways

3. **(More Movies)** Mary and 4 friends go to a movie. How many ways can they sit next to each other with Mary always between two friends?

Ans: The total number of ways they can sit together is $5! = 5\times4\times3\times2\times1 = 120$. The total number of ways they can sit together with Mary on the left is 4!, which is the same as if Mary sat on the right. Hence, the total number with Mary between two friends is $5! - 4! - 4! = 120 - 24 - 24 = 72$ ways

4. **(Baseball Season)** A baseball league consists of 6 teams. How many games will be played over the course of a year if each team plays every other team exactly 5 times?

Ans: $5\binom{6}{2} = 5\frac{6!}{2!4!} = 75$

5. **(Distinct 3-digit numbers)** How many 3-digit numbers are there in which the number of even digits is 0,2,4,6, or 8 ?

Ans: Calling even digits by E and odd digits by O, we can have any 3-digit numbers of the form EEO, EOE, OEE, and OOO. Now there are 5 even digits 0,2,4,6, 8 and there are 5 odd digits 1,3,5,7,9. However, the number 0 cannot start the 3-digit number, else the number is a 2-digit number. Hence, using the multiplication principle, we have

EEO: $4\times5\times5 =$ 100 3-digit numbers of this form
EOE: $4\times5\times5 =$ 100 3-digit numbers of this form
OEE: $5\times5\times5 =$ 125 3-digit numbers of this form

for a total of 450 possibilities.

6. **(Interesting Problem)** How many 3-digit numbers $d_1d_2d_3$ are there whose digits add up to

8? That is, $d_1 + d_2 + d_3 - 8$, Note: 063 is not a 3-digit number, it is the 2-digit number 63.

Ans: Starting with a 3-digit number $d_1d_2d_3$, we start with the hundreds digit d_1, which can have values $k = 1,2,3,4,5,6,7$ or 8 for a total of 8 possibilities (0 is not a possibility for the hundreds digit else $d_1d_2d_3$ will only be a 2-digit number. If we now pick the hundreds digit to be k, then the number of possibilities for the 10-digit d_2 will be $8-k+1$ (we need the +1 since now 0 is a possibility). The 1-digit d_3 is now determined and so if we add up these possibilities of the hundred and ten digit we find

$$\begin{aligned}\sum_{k=1}^{8}(8-k+1) &= \sum_{k=1}^{8}9-\sum_{k=1}^{8}k \\ &= 9\cdot 8-\frac{8\cdot 9}{2} \\ &= 72-36 \\ &= 36 \text{ ways}\end{aligned}$$

7. **(World Series Time)** Two teams are playing in playoff series. If it is a best of 3-game series there are three possible outcomes, WW, WLW, LWW if we do not distinguish the teams. If the teams are distinguished, say team A and team B, then there would be six outcomes in a 3-game series, AA, ABA, BAA, BB, BAB, ABB.

a) How many possible games are there is a 5-game series if we do not distinguish the teams?

b) How many possible games are there is a $2n+1$ game series if we do *not* distinguish the teams?

Ans: a) If we denote the possible games as

1 2 3 4 5

the three W's can be placed in any 3 slots. Hence the number of possible games is

$$\binom{5}{3}=10$$

c) The W's can be placed in any $n+1$ of the $2n+1$ possible slots. Hence, the number of possible games is

$$\binom{2n+1}{n+1}.$$

8. **(Bell Numbers)** A partition of a set A is defined as asset of nonempty subsets, pairwise disjoint, whose union is A. The number of such partitions of a set of size n is called the Bell number B_n. For example when $n=3$ the Bell number $B_3=5$ since there are 5 partitions of a set of size 3. For example, if $A=\{a,b,c\}$ the 5 partitions are:

$$\{\{a\},\{b\},\{c\}\}$$
$$\{\{a\},\{b,c\}\}$$
$$\{\{b\},\{a,c\}\}$$
$$\{\{c\},\{a,b\}\}$$
$$\{\{a,b,c\}\}$$

Enumerate the partitions of the set $\{a,b,c,d\}$ to find the Bell number B_4.

Ans: $B_4 = 15$

9. **(Two Committees)** How many ways can the Snail Darter Society, who has 25 members, elect an executive committee of 2 members and an entertainment committee of 4 members if no member of the society can serve on both committees?

Ans: $\binom{25}{2}\binom{23}{3} = \left(\frac{25!}{23!2!}\right)\left(\frac{23!}{20!3!}\right) = 300\times 1771 = 531{,}300$

10. **(Serving on More than One Committee)** How many ways can the Snail Darter Society, who has 25 members, elect an executive committee of 2 members, an entertainment committee of 3 members, and a welcoming committee of 2 members if members can serve on more than one committee?

Ans:

$$\binom{25}{2}\binom{25}{3}\binom{25}{2} = \left(\frac{25!}{23!2!}\right)\left(\frac{23!}{20!3!}\right)\left(\frac{25!}{23!2!}\right)$$
$$= 300\times 2300\times 300$$
$$= 207{,}000{,}000$$

11. **(Counting Softball Teams)** A college softball team is taking 25 players on a road trip. The traveling squad consists of 3 catchers, 6 pitchers, 8 infielders, and 8 outfielders. Assuming each player can only play her own position, how many different teams can the coach put on the field?

Ans:

$$\binom{3}{1}\binom{6}{1}\binom{8}{4}\binom{8}{3} = 3\times 6\times 70\times 56 = 70{,}560 \text{ teams}$$

In other words they could field a different team every day for over 193 years.

12. **(Single-Elimination Tournaments)** In a 64-team single elimination tournament, what is the total number of possible outcomes of the tournament?

Ans: The total number of games played is 63 since there is a 1-1 correspondence between the ultimate losers in the tournament (which is 63) and the game ball of the tournament. Since each game has 2 possible outcomes, the total number of outcomes of the entire tournament is $2^{63} = 9223372036854775808$.

13. **(Permutations as Groups)** The $3! = 6$ permutations of set $\{1,2,3\}$ can be illustrated by the 2×3 arrays

$$\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \qquad \gamma = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$
$$\delta = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \qquad \eta = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

where the bottom row of each array shows how the top row is permuted. Carrying out one permutation followed by another defines a multiplication of the permutations. For instance $\varepsilon = \delta\beta$ means we do permutation β first then permutation δ second, which yields the permutation ε (check it yourself). Compute the following.

a) $\alpha\beta$

Ans: $\alpha\beta = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \alpha$

b) α^2

Ans: $\alpha^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \alpha$

c) β^2

Ans: $\beta^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \gamma$

d) $\delta\varepsilon$

Ans: $\delta\varepsilon = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \gamma$

After you get the hang of multiplying permutations, make a 6×6 multiplication table of all products. This table describes what in group theory is called the **symmetric group** of order 3, denoted by S_3.

14. **(The Josephus Problem)** A thousand Roman slaves are put in a circle numbered from 1 to 1000. All are to be shot except one lucky survivor. Slave #1 graciously allows slave #2 to begin, so the order of shooting is 2,4,6,… . After a slave is shot,

the body is removed from the circle and the shooting continues. For example, if there were six slaves the order of the shooting would be 2,4,6,3,1 and slave #5[1].

a) Find the position $p(n)$ of the surviving slave if there are n slaves for $n = 1, 2, \ldots, 10$.

Ans:

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	**1**	**1**	**3**	**1**	**3**	**5**	**7**	**1**	**3**	**5**

b) Verify that if n is odd, $p(n)$ satisfies the recurrence relation $p(2n+1) = 2p(n)+1,\ p(1) = 1$.

Ans:

$$p(3) = 2p(1)+1 = 2\cdot 1+1 = 3$$
$$p(5) = 2p(2)+1 = 2\cdot 1+1 = 3$$
$$p(7) = 2p(3)+1 = 2\cdot 3+1 = 7$$
$$p(9) = 2p(4)+1 = 2\cdot 1+1 = 3$$

c) Verify that if n is even, $p(n)$ satisfies the recurrence relation $p(2n) = 2p(n)-1,\ p(2) = 1$.

Ans:

$$p(2) = 2p(1)-1 = 2\cdot 1-1 = 1$$
$$p(4) = 2p(2)-1 = 2\cdot 1-1 = 1$$
$$p(6) = 2p(3)-1 = 2\cdot 3-1 = 5$$
$$p(8) = 2p(4)-1 = 2\cdot 1-1 = 1$$
$$p(10) = 2p(5)-1 = 2\cdot 3-1 = 5$$

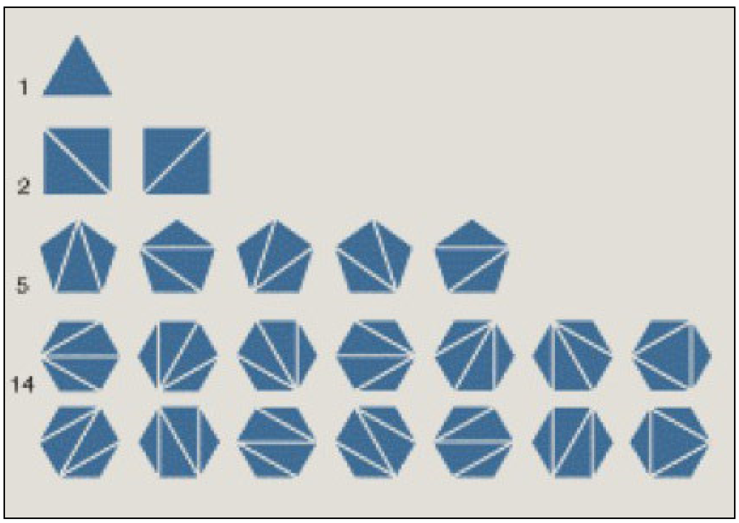
d) Which position would you choose if there were 100 slaves in the circle?
Ans: We can use the results from b), c), getting

[1] The problem is named after Flavius Josephus, a Jewish historian living in the 1st century. According to legend, he and 40 fellow soldiers were trapped in a cave, surrounded by Romans. They chose suicide over capture and decided that to form a circle and start killing themselves, killing alternate persons in the circle.

$$
\begin{aligned}
p(3) &= 2p(1)+1 = 2\cdot 1+1 = 3 \\
p(6) &= 2p(3)-1 = 2\cdot 3-1 = 5 \\
p(12) &= 2p(6)-1 = 2\cdot 5-1 = 9 \\
p(25) &= 2p(12)+1 = 2\cdot 9+1 = 19 \\
p(50) &= 2p(25)-1 = 2\cdot 19-1 = 37 \\
p(100) &= 2p(50)-1 = 2\cdot 37-1 = 73
\end{aligned}
$$

Hence, slave #73 is the lucky one.

15. **(Catalan Numbers)** The Catalan number $C_n, n = 1,2,3,\ldots$ is the number of different ways a convex polygon with $n+2$ sides can be subdivided into triangles. The following figure shows the Catalan numbers $C_1, C_2, C_3,$ and C_4. Can you find the Catalan number C_5?



First four Catalan numbers

Ans: 42. The general formula for the number of ways c_n to dissect a convex polygon is

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

16. **(Famous Apple Problem)** We wish to distribute 8 identical apples to 4 children. How many ways is this possible if each child gets at least one apple?

Ans: Think of the 8 apples in a row like a a a a a a a a . We can distribute the apples to the 4 children by placing 3 bars | in some of the 7 gaps between the apples. For example, if we place the bars like a a | a | a a a a | a

this means we give the first child 2 apples, the second child 1 apple, the third apple 4 apples, and the 4th child 1 apple. So the total number of ways to do this is the number of ways of selecting the 3 bars from 7 gaps, or

$$\binom{7}{3} = \frac{7!}{3!4!} = 35 \text{ ways} .$$

19. **(Derangements)** A derangement is a permutation in which none of the elements remain in their natural order. For example the only derangements of $(1,2,3)$ are $(3,1,2)$ and $(2,3,1)$. Hence we write $!3 = 2$. Nicolas Bernoulli proved that the number of derangements of a set of size n is

$$!n = n! \sum_{k=1}^{n} \frac{(-1)^k}{k!}$$

How many derangements are there for the members $(1,2,3,4)$? Enumerate them.

Ans: We begin by putting B,C,D in the first position. If B is in the first position we put A,C,D in the second position, and so on. Continuing in this manner we see that there are 9 derangements of 4 objects, which are : BADC, BCDA, BDAC, CADB, CDAB, CDBA,DABC, DCAB, DCBA

20. **(Counting Handshakes)** There are 20 people in a room and each person shakes hands with everyone else. How many handshakes are there?

Ans: The first person shakes hands with 19 other persons, the second person shakes hands with 18 other persons (since we have already counted the handshake with the first person). Continuing on, the third person shakes hands with 17 persons, etc … so the total number of handshakes is

$$19 + 18 + 17 + \ldots + 2 + 1 = \frac{19 \times 20}{2} = 190 \text{ handshakes}$$

21 **(Counting Functions)** How many functions are there from $A = \{a,b,c\}$ to $B = \{0,1,2\}$? Write them down and draw the graphs for a few of them.

Ans: The value of the function at a can be either 0 or 1; likewise the value of the function at b can be 0 or 1; likewise at c. Hence, the total number of functions is $2^3 = 8$. The eight functions, calling them all by f, are

$$\begin{gathered}
f(a)=0, f(b)=0, f(c)=0 \\
f(a)=0, f(b)=0, f(c)=1 \\
f(a)=0, f(b)=1, f(c)=0 \\
f(a)=0, f(b)=1, f(c)=1 \\
f(a)=1, f(b)=0, f(c)=0 \\
f(a)=1, f(b)=0, f(c)=1 \\
f(a)=1, f(b)=1, f(c)=0 \\
f(a)=1, f(b)=1, f(c)=1
\end{gathered}$$

22 **(Combinatorial Euclidean Algorithm)** Finding the greatest integer that divides two positive integers can be interpreted as a combinatorial problem where one seeks all possibilities of divisors. However, the method of finding the greatest common divisor, denoted $\gcd(n,m)$, of two natural numbers n,m is called the **Euclidean Algorithm** and avoids the combinatorial headache of looking through all divisors. The algorithm makes the fundamental observation that the quotient of m divided by n has the form:

$$\frac{n}{m} = q + \frac{r}{m}\ ,\ 0 \le r < m$$

where q is the quotient of n/m and r the remainder. Using the basic property[2] $\gcd(n,m) = \gcd(m,r)$, it is possible to find a sequence of "decreasing" gcds, which eventually lead to an obvious gcd. For example, the greatest common divisor of 255 and 68 is 17 as seen from the "decreasing" sequence of gcds:

$$\gcd(255,68) = \gcd(68,51) = \gcd(51,17) = \gcd(17,0) = 17$$

The numbers 51, 17, 0 in the gcds are the remainders of

[2] This is true since any whole number that divides both m and n also divides n and $r = n - mq$, and vice versa. true.

remainder of $\frac{255}{68}$ is 51
remainder of $\frac{68}{51}$ is 17
remainder of $\frac{51}{17}$ is 0

23. Use the Euclidean Algorithm to find the greatest common divisors of the following pairs of numbers[3].

a) $\gcd(25,3)$
b) $\gcd(54,36)$
c) $\gcd(900,22)$
d) $\gcd(816,438)$

Ans:

a) $\gcd(25,3)=1$
b) $\gcd(54,36)=1$
c) $\gcd(900,22)=6$
d) $\gcd(816,438)=6$

24. **(Partitions)** A partition of a natural number n is a sequence of natural numbers $a_1 \geq a_2 \geq \cdots$ such that $n = a_1 + a_2 + \cdots + a_k$ for some k. There are two partitions of 2, which are $2=2$, $2=1+1$, which we denote by 2, 11. There are three partitions of 3, which are $3=3$, $3=2+1$, $3=1+1+1$, which we denote by 3, 21, 111. Find the partitions of 4, 5 and 6. How would you construct a total order for the partitions of the natural numbers? You are relieved from finding the partitions of $n=200$ since there are 3,972,999,029,388 of them.

Ans: The partitions of 4 are 1111, 211, 22, 31, and 4 . The partitions of 5 are 11111, 2111, 221, 311, 32, 41, 5 . There are 11 partitions of 6 , which are 111111, 21111, 2211, 3111, 222, 321, 411, 33, 42, 51, 6.

A total order "$\prec$" for a partition is $a_1 + a_2 + \cdots \prec b_1 + b_2 + \cdots$ if and only if $a_1 + a_2 + \cdots + a_k \leq b_1 + b_2 + \cdots + b_k$ for all $k \geq 1$. With this ordering we have

[3] There are other methods for finding the greatest common divisor of two natural numbers. One could factor each natural number into its prime components, find the common factors, then take the product of the common factors. For example, 816 and 438 have factors of 2 and 3, and so 6 would be their greatest common divisor.

$n = 4 : 1111 \prec 211 \prec 22 \prec 31 \prec 4$

$n = 5 : 11111 \prec 2111 \prec 221 \prec 311 \prec 32 \prec 41 \prec 5$

$n = 6 : 11111 \prec 21111 \prec 2211 \prec 222 \prec 3111 \prec 321 \prec 33 \prec 411 \prec 42 \prec 51 \prec 6$

25 **(Euler's Theorem)** There are many identities related to integer partitions. The first one was due to the Swiss mathematician Leonhard Euler who proved that any number has as many partitions consisting of odd numbers as there are partitions consisting of distinct numbers. For example, the number 5 has 3 partitions consisting only of odd numbers, 11111, 311, and 5; and 3 partitions consisting of distinct numbers is 5, 41, 32. Show Euler's partition identity holds for the number 6.

Ans: Partitions consisting of odd numbers: 111111, 3111, 33, 51 and partitions of distinct numbers: 321, 42, 51, and 6.

17. **(Fun Problem)** How many integers are there between 000 and 999 where the middle digit is the average of the other two digits? For example, 246 and 420 satisfy the condition.

Ans: Firet make the observation that the first two digits cannot be 0 and that the only possible numbers where the last digit is 0 are the four numbers 210, 420, 630, and 840. We now include the nine numbers 111, 222, 333, … , 999 where all the digits are the same. We now list the remaining possibilities when the middle digit is 2,3,4,… 8. We have

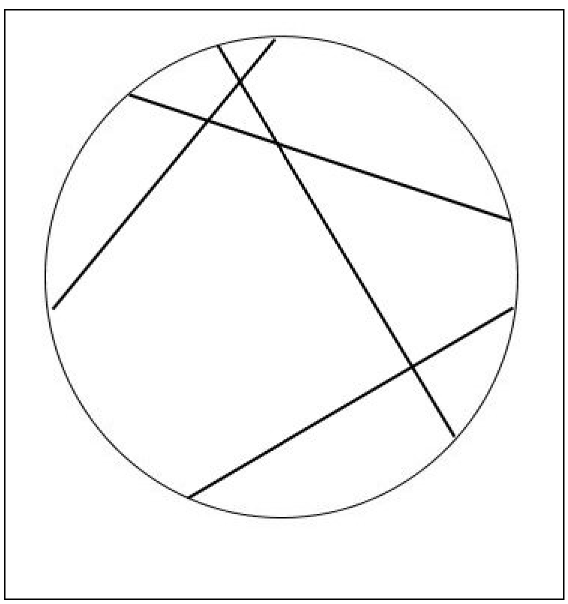
middle digit 2: 123
middle digit 3: 135, 234
middle digit 4: 147, 246, 345
middle digit 5: 159, 258, 357, 456
middle digit 6: 369, 468, 567
middle digit 7: 579, 678
middle digit 8: 789

and

middle digit 2: 321
middle digit 3: $531, 432$
middle digit 4: $741, 642, 543$
middle digit 5: $951, 852, 753, 654$
middle digit 6: $963, 864, 765$
middle digit 7: $975, 876$
middle digit 8: 987

Counting up these we have $2(16)+9+4=45$. Now, if you want to be clever, realize that that the first and last digits must both be odd or both even. Thus there are $5 \cdot 5 = 25$ possible numbers where the first and last digits are odd, and $5 \cdot 4 = 20$ even-even pairs that do not use 0 in the first digit. Hence we have a total of 25 + 20 = 45 numbers.

27. **(Basic Pizza Cutting)** You are given a circular pizza and ask to cut it into as many pieces as possible, where a cut means any line that passes all the way through the pizza, although not necessarily through the center. The picture below shows a pizza with four typical slices dividing the pizza into 9 pieces of pizza.[4]



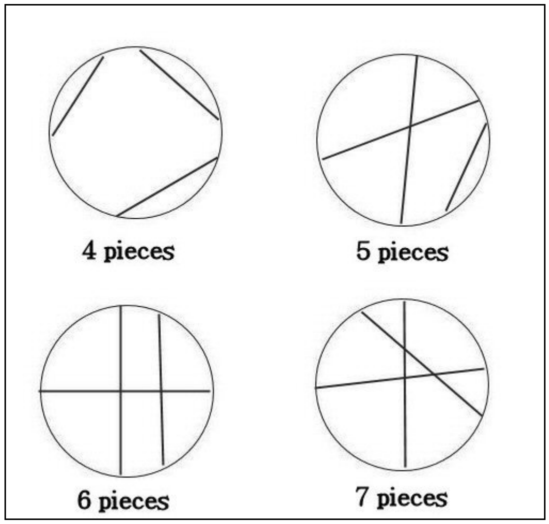
Typical pizza with 4 cuts and 9 pieces

a) How many different pieces can you obtain with three slices?

[4] We could just as well said that the entire plane can be subdivided into 9 disjoint sections. In fact normally the "pizza problem" is stated in terms of the entire plane and not simply a the inside of a circle.

b) How many different pieces can you obtain with four slices?
c) Show how you can slice a pizza with 4 slices to obtain 5, 6, 7, 8, 9, 10, and 11 pieces ?

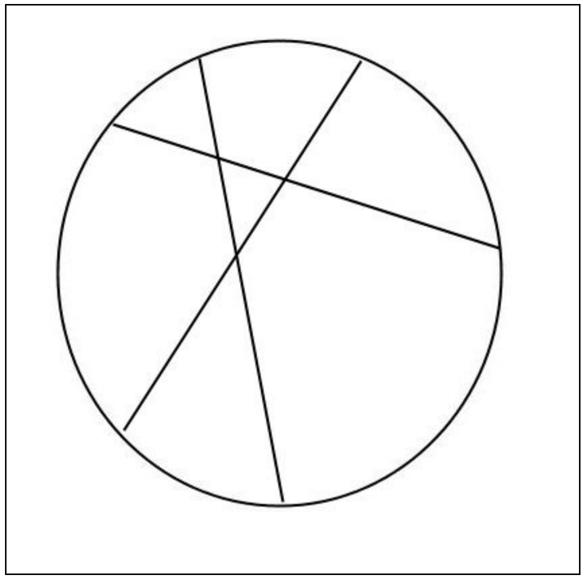
Ans: a) With 3 slices one can obtain 4, 5, 6, or 7 pieces. If none of the slices intersect then there will be 4 slices, if two slices intersect there will be 5 slices, etc. Examples are



b) With 4 slices one can obtain 5-11 pieces, depending on the number of intersections of the slices, assuming the intersections only involve two slices. We leave this fun problem to the reader. (Note that if all the slices are all made through the center of the pizza, then the number of pieces of pizza will be twice the number of slices.)

c) We leave this fun problem for the reader.

28. **(Pizza Cutter's Formula)** A pizza cutter wants to cut a pizza in such a way that it has the maximum number of pieces, not necessarily the same size or shape. The only restriction on how the pizza is cut is that each cut much pass all the way through the pizza, not necessary through the center.



Typical Pizza after 3 cuts with 7 pieces

a) If $P(n)$ is the maximum number of pieces from n cuts, then $P(n) = P(n-1) + n$.

b) Show the pizza cutter's formula is

$$P(n) = \frac{n^2 + n + 1}{2} .$$

Ans: a) The first line cuts the pizza into 2 pieces, the second piece into a max of 4 pieces and the third line yields a maximum of 7 pieces, and so on. We made the nth slice so it cuts the previous $n-1$ slices (we can do this by making it very close to the previous slice), hence the nth slice is cut into n segments, each of those segments cutting one piece of pizza into two pieces, and so the nth slice yields n new pieces of pizza. Hence, if $P(n-1)$ is the maximum number of pieces after $n-1$ cuts, then to maximize the number after n cuts we make the n-th cut intersect each of the $n-1$ cuts, giving an extra n pieces. Hence $P(n) = P(n-1) + n$.

b) We can write

$$\begin{aligned}
P(n) &= P(n-1)+n \\
&= \left[P(n-1)+(n-1)\right]+n \\
&= \left[P(n-2)+(n-1)\right]+(n-1)+n \\
&\qquad \cdots \qquad\qquad \cdots \\
&= \left[P(1)+0\right]+1+2+\cdots+n \\
&= 1+(1+2+\cdots+n) \\
&= 1+\frac{n(n+1)}{2} \\
&= \frac{n^2+n+2}{2}
\end{aligned}$$

Note that for $n \geq 3$ this is larger than if the pizza cutter made all cuts through the *center* of the pizza. If the cuts are made through the center of the pizza, then the number of pieces is twice the number of cuts. Cutting a pizza into pieces is the same as cutting the plane into pieces. More problems along this line is to cut 3-dimensional space by planes, and higher-dimensional spaces by hyper-planes. The reader can find more about how to divide spaces by lines, circles, ellipses, etc by "googling" phrases like "subdividing the plane into regions."

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