Problems 2.4 Countable Infinity

1. Which of the following sets S are finite? For finite sets, find if possible, the cardinality of the sets.

a) S is any set that can be put in a one-to-one correspondence with a proper subset of itself.

Ans: infinite, in fact that is one of the definitions of an infinite set.

b) $\{a,b,c,d\}$

c) Consider finite strings of the letters a,b such as $ab,\ a,\ bbbaaaa, aba, baaaabb,$ etc. The set S is the set of all finite strings of the letters a,b.

Ans: S constains an infinite number of elements, some of them are $\{a, aa, aaa, aaaa, aaaaa, \dots\}$ which are all finite strings but the set of strings contains an infinite number of them.

d) the set of rational numbers

Ans: infinite, but what kind of infinity. We will see later that the set of irrational numbers is also infinity but a larger infinity.

e) the collection of five card hands dealt from a deck of 52 cards

Ans: finite, $\binom{52}{5} = 2{,}598{,}960$ to be exact

f) prime numbers greater than 10^{10}

Ans: infinite, although there are a finite number of prime numbers *less* than 10^{10}, and the prime number theorem states there are approximately

$$\frac{10^{10}}{\ln 10^{10}} \doteq 434{,}294{,}145$$

prime numbers less than 10^{10}.

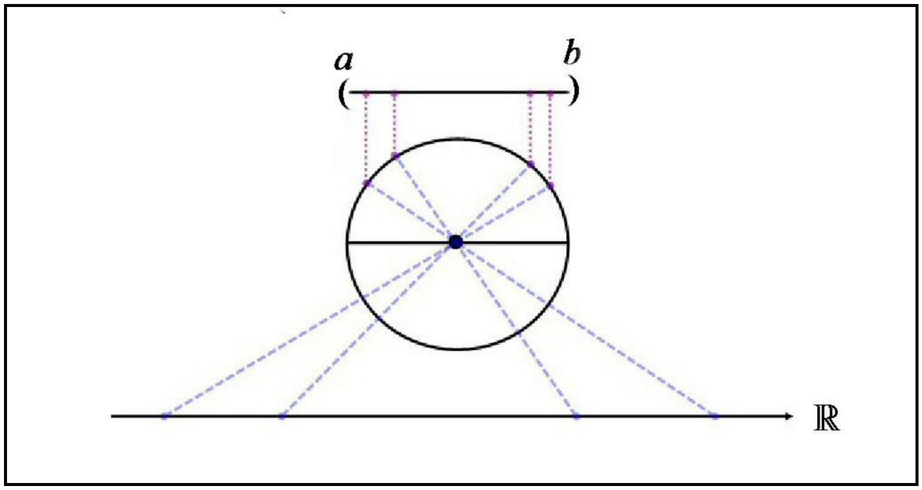
g) $\{n \in \mathbb{N} : n^2 \text{ is odd}\}$

Ans: infinite, they are the odd integers $1,3,5,\dots$

h) $\{n \in \mathbb{N} : n \text{ is even and prime}\}$

Ans: finite, the only even prime number is 2

2. Explain why the following figure provides a geometric proof that the real line is equivalent to any open interval.



Ans: There is a 1-1 association between points $x \in (a,b)$ and points $y \in \mathbb{R}$ as follows. For each $x \in (a,b)$ a unique $y \in \mathbb{R}$ is found by moving vertically downward on the dotted line from x until one reaches the drawn circle, and then along the dotted line that passes through the center of the circle until one reaches the point x on the real line $\mathbb{R}$. Note that as the point x gets closer to the end points a and b the image point y will approach plus or minus infinity respectively on the real line.

3. Show that the union of two countable sets is countable.

Ans: If A, B are countable, then their elements can be put in a 1-1 correspondence with the natural numbers and so we can denote them by $A = \{a_1, a_2, \ldots\}, B = \{b_1, b_2 \ldots.\}$. The union of these countable sets can then be expressed as $A \cup B = \{a_1, b_1, a_2, b_2 \ldots\}$, which can put in a 1-1 correspondence with the natural numbers by

$$
\begin{aligned}
&1 \approx a_1 \\
&2 \approx b_1 \\
&3 \approx a_2 \\
&4 \approx b_2 \\
&\quad \ldots \\
&2n-1 \approx a_n \\
&2n \approx b_n \\
&\quad \ldots
\end{aligned}
$$

Hence, the union of countable sets is again countable.

4.**(Equivalent Sets)** For the following intervals, find an explicit one-to-one correspondence that show the intervals are equivalent.

a) $\{a,b,c\} \approx \{1,2,3\}$

Ans: $a \approx 1,\ b \approx 2,\ c \approx 3$. Since each set can be put in a 1-1 correspondence with the set $\{1,2,3\}$ we say the sets have cardinality 3.

b) $A = [0,1) \approx B = [0,\infty)$

Ans: Consider the 1-1 correspondence $f : A \to B$, where $f(x) = \dfrac{x}{1-x}$. We show this function is a bijection from $[0,1)$ onto $[0,\infty)$ by first showing f is 1-1. We do this by setting

$$f(a) = \frac{a}{1-a} = \frac{b}{1-b} = f(b)$$

which implies $a(1-b) = b(1-a)$ or $a - ab = b - ab \Rightarrow a = b$. Hence f is 1-1. We now show f is onto $[0,\infty)$ by taking an arbitrary number $b \in [0,\infty)$ and showing $\exists a \in [0,1)$ such that $f(a) = \dfrac{a}{1-a} = b$. But this equation yields $a = b(1-a) \Rightarrow b - ab$, and solving for a gives $a = \dfrac{b}{1+b} \in [0,1)$. Hence f is a bijection from $[0,1) \to [0,\infty)$.

c) $(0,1) \approx \mathbb{R}$

Ans: The function $f(x) = \tan(\pi x),\ 0 < x < 1$ is a bijection from $(0,1) \to \mathbb{R}$.

d) $[0,1] \approx [3,5]$

Ans: The linear function $f(x) = 2x + 3$ is a bijection from $[0,1] \to [3,5]$

5. **(Cardinality of the Power Set)** Let $A = \{a,b,c,d\}$ and consider the mapping $f : A \to P(A)$ from the set into its power set, defined by

$$f(a) = \{a,c,e\}.\ f(b) = \{d\},\ f(c) = \{a,b\}.\ f(d) = \{d\}$$

Using the rule in Theorem 1, construct a subset of A that is different from any of the images of f.

Ans: We select those members of A that do not belong to their image under f, which are the members b and c. Hence extra set $= \{b, c\}$

6. **(Cantor-Bernstein Theorem)** Given any two sets A and B, suppose that a subset of A is equivalent to B and that a subset of B is equivalent to A. Then A and B are equivalent. Use this theorem to prove $(0,1) \approx [0,1]$.

Ans: The set $(0,1)$ is equivalent to the subset $(0,1) \subseteq [0,1]$ by the identity mapping, and the set $[0,1]$ is equivalent to the subset

$$\left[\frac{1}{4}, \frac{3}{4}\right] \subseteq (0,1)$$

using the bijection

$$f(x) = \frac{x}{2} + \frac{1}{4}.$$

Hence, by the Cantor-Bernstein theorem the open interval $(0,1)$ has the same cardinality as the closed interval $[0,1]$.

7. **(Even and Odd Natural Numbers)** Let E be the set of even positive integers and O be the set of odd positive integers. Given an explicit function to show the following equivalences.

a) $E \approx O$ **Ans:** $2n-1 \approx 2n,\ n = 1, 2, \ldots$

b) $\mathbb{N} \approx O$ **Ans:** $n \approx 2n-1,\ n = 1, 2, \ldots$

c) $N \approx E$ **Ans:** $n \approx 2n, n = 1, 2, \ldots$

d) $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$

Ans: $1 \leftrightarrow (1,1), 2 \leftrightarrow (1,2), 3 \leftrightarrow (2,1), 4 \leftrightarrow (1,3), 5 \leftrightarrow (2,2), 6 \leftrightarrow (3,1)$, etc

8. **(Infinite Sets)** A set if infinite if and only if it is equivalent to a proper subset of itself. Use this definition of an infinite set to show the following sets are infinite.

a) $\mathbb{N}$

Ans: The set of natural numbers $\mathbb{N}$ is equivalent to the even natural numbers. A bijection is $f(n) = 2n,\ n = 1, 2, \ldots$.

b) $\mathbb{R}$

Ans: $\mathbb{R}$ is equivalent to the subset $(0,1) \subseteq \mathbb{R}$. A bijection $f : (0,1) \to \mathbb{R}$ is $f(x) = \tan(x)$.

9. **(Bijection from the Prime Numbers)** The following sets are equivalent. Find a bijection
$f : A \to B$.

a) $A = \mathbb{N},\ B =$ prime numbers natural numbers and the prime numbers

Ans: $f(n) = nth$ prime number ; $1 \leftrightarrow 2,\ 2 \leftrightarrow 3,\ 3 \leftrightarrow 5,\ 4 \leftrightarrow 7, \ldots$

b) $A = \mathbb{N}, B = \{10, 12, 14, \ldots\}$

Ans: $n \leftrightarrow 8 + 2n$

c) $A = \mathbb{R}, B = (0, \infty)$

Ans: $f(x) = e^x$

10. **(Interesting Equivalence)** Show $(0,1) \approx (0,1]$.

Ans: Although we could use the Cantor-Bernstein theorem as discussed in Problem 5, we will give a direct proof. Although the set $(0,1]$ has only 1 more element than $(0,1)$, it is not a trivial matter to show the sets are equivalent. Here is a clever way to do it. Set up a 1-1 correspondence f from $(0,1] \to (0,1)$ as follows. Let

$$S = \left\{ \frac{1}{n+1} : n = 1, 2, \ldots \right\}$$

be a sequence of points in $(0,1)$ and define the a function f by

$$f(0) = 1/2$$
$$f(1) = 1/3$$
$$f(x) = \begin{cases} \dfrac{x}{2x+1} & x \in S \subseteq (0,1) \\ x & x \notin S \subseteq (0,1) \end{cases}$$

This function is a bijection from $(0,1] \to (0,1)$. Looking more carefully at the function we see

$$f(0) = 1/2$$
$$f(1) = 1/3$$
$$f(1/2) = 1/4$$
$$f(1/3) = 1/5$$
$$f(1/4) = 1/6$$
$$f(1/5) = 1/7$$
$$\dots$$

Draw a picture of the sets $(0,1]$ and $(0,1)$ and visualize how the numbers in the two gets get matched up.

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