Problems 2.5 Uncountable Sets

1. **(Visual Correspondences)** Construct visual correspondences between the following sets to show they are equivalent.

a) two circles of different radii

Ans: Center both circles at the original and draw rays from the origin. Each ray will create a 1-1 correspondence between the circles.

b) $(0,\infty)$ and $(-\infty,0)$

Ans: Draw a family of semi-circles in the upper half plane centered at the origin. The two endpoints where the semi-circles touch the x-axis create a 1-1 correspondence $x \leftrightarrow -x$ for each $x > 0$.

c) $S = (-1,1)$ and $\mathbb{R}$

Ans: First place a lower semi-circle of radius 1 centered at $x = 0, y = 1$ in the Cartesian plane, then vertical lines from points in the interval (-1,1) on the x-axus to points on the semi-circle represent a 1-1 correspondence. Then, draw rays from $x = 0, y = 1$ to points on the x-axis. This is a 1-1 correspondence between the points on the lower semi-circle and real numbers, and hence points in the interval $(-1,1)$ on the x-axis and the entire x-axis. Combining these two 1-1 correspondences gives the desired 1-1 correspondence.

d) The points on a sphere minus the top point and points in the plane.

Ans: Draw rays from the top point of the sphere to points in the plane. The intersection of the rays with the sphere and the points in the plane give a 1-1 correspondence. This mapping of the sphere to the plane is called the sterographic projection of the sphere (minus the north pole).

2. **(Cardinality of Functions)** Show that the cardinality of the set of functions $F = \{f : \mathbb{N} \to \mathbb{N}\}$ is uncountable.

Ans: We show that the set of functions $f : \mathbb{N} \to \mathbb{N}$ can not be put in a 1-1 correspondence with the natural numbers $\mathbb{N}$.The proof is by contradiction. That is, suppose there *is* a 1-1 correspondence from $\mathbb{N}$ to $F : 1 \to f_1,\ 2 \to f_2,\ 3 \to f_3,\ \ldots$ But no matter what the correspondence, we can construct a function $g \in F$ that is not in the list $f_1, f_2, \ldots$ How do we do this? We choose g so that

$g(1)$ so it is different from $f_1(1)$
$g(2)$ so it is different from $f_2(2)$
$g(3)$ so it is different from $f_3(3)$
... ...

In this way the function $g \in F$ is different from all the functions $\{f_n : n = 1,2,\dots\}$ since it differs from each f at at least one point. Hence, the cardinality of the set of functions from the natural numbers to the natural numbers is not countable, hence uncountable.

3. **(Irrational Numbers)** Show that the irrational numbers in the interval $[0,1]$ is uncountable.

Ans: The set $[0,1]$ is uncountable and the rational numbers is countable. Hence, if we assume the irrational numbers were countable, then the set $[0,1]$ would be countable since $[0,1]$ would be the union of two countable sets, rational and irrational numbers in $[0,1]$. But $[0,1]$ is uncountable and hence we must assume the irrational numbers is an uncountable set.

4. **(Bijection from the Unit Square to $\mathbb{R}^2$)** Let

$$S = \{(x, y) : 0 < x < 1,\ 0 < y < 1\}$$

show that the function $F : S \to \mathbb{R}^2$ defined by

$$F(x, y) = (f(x), f(y))$$

where

$$f(x) = \tan\left[\pi\left(x - \frac{1}{2}\right)\right],\ 0 < x < 1$$

is a bijection from S to $\mathbb{R}^2$. Evaluate the function F at various points inside the unit square S to get a feel for the function.

5. **(Visual Equivalence)** Let S denote the unit square with vertices $(1,1),(-1,1),(-1,-1),(1,-1)$, and let

$$C = \{(x, y) : x^2 + y^2 = 1\}$$

denote the unit circle. Show these sets are equivalent with some type of visual drawing.

Ans: Draw rays from the origin. Each ray will intersect both the square and circle exactly once giving a 1-1 correspondence.

6. **(Algebraic Numbers)** Show that the numbers

$$\frac{1+\sqrt{3}}{2} \text{ and } \frac{1-\sqrt{3}}{2}$$

are algebraic numbers.

Ans: They are the roots of the equation $x^2+x+1=0$

7**. (Transcendental Numbers)** Some famous transcendental numbers are $\pi, e, e^{\pi}, \pi^e, \ln 2, \ldots$ .The first provable transcendental number was found by the French mathematician Joseph Louiville (1809-1882) in 1844, who constructed and proved the constant

$$L=\sum_{n=1}^{\infty} 10^{-n!}$$

is transcendental. Write out the first few decimal digits of this number. Google "Liouville's
constant" and read more about it.

Ans: $L=\sum_{n=1}^{\infty} 10^{-n!}=0.110001000000000000000001\ldots$ The only nonzero digits are in the 1^{st},
2^{nd}, 6^{th}, 24^{th}, 120^{th},… (n!th) decimal place. elsewhere.

8. Prove that if we add a member to a denumerable set A, we still have a denumerable set.

Ans: If A is denumerable, we denote it as $A=\{a_1,a_2,\ldots\}$. If we no add a new member we can write

$$A=\{a_1,a_2,\ldots\}\cup\{a\}=\{a,a_1,a_2,\ldots\}$$

which is equivalent to A by means of the 1-1 correspondence $a_1 \leftrightarrow a,\ a_{n+1} \leftrightarrow a_n,\ n=1,2,\ldots$.

8. ($\mathbb{R} \approx \mathbb{R}^3$) Show that the unit cube

$$C=\{(x,y,z): x,y,z\in\mathbb{R},\ 0<x<1,\ 0<y<1,\ 0<z<1\}\subseteq\mathbb{R}^3$$

is equivalent to the unit interval $(0,1)\subseteq\mathbb{R}$. Hint: Use the technique Cantor used to prove $(0,1)$
is equilvalent to the unit square.

Ans: The mapping

$$f:(0.a_1a_2a_3\ldots,\ 0.b_1b_2b_3\ldots,\ 0.c_1c_2c_3\ldots)\to(0.a_1b_1c_1a_2b_2c_2\ldots)$$

is a bijection from the unit cube in three dimensions to the unit interval $(0,1)$. One can continue this argument to show that all the higher dimensional spaces $\mathbb{R}^n$ are equilvalent to the real line.

10. **(Picking off Members from an Infinite Set)** Suppose you have an infinite set of any infinite size, $\aleph_0, c, 2^c, \ldots$ and suppose you to judiciously pick off members one at a time from the set, say 1, 2, 3, ….you never run out of numbers. Calling this set of numbers you have picked off A, we arrive at the conclusion that A any infinite set has a subset. What can you conclude from this observation?

Ans: Since a countable set is a subset of *any* infinite set and since $A \subseteq B \Rightarrow |A| \leq |B|$ we conclude that $\aleph_0$ is the smallest infinity.

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