Problems 2.6 Larger Infinities and the ZFC Axioms

1. **(Cardinality of Sets of Functions)** Show that the set of all functions defined on the natural numbers with values 0 and 1 has cardinality c. Hint: Relate each sequence of 0's and 1's to a subset of natural numbers and then use Cantor's theorem.

Ans: For every function $f:\mathbb{N}\to\{0,1\}$ corresponds to a subset of the natural numbers. For example the function defined by

$$f(1)=0,\ f(2)=1, f(3)=1,\ f(n)=0,\ n\geq 4$$

would correspond to the subset $\{2,3\}\in P(N)$. In general, if the function f is 1 for some natural number, then that natural number is included in the subset. If the function f is 0 at some natural number, then that natural number is not included in the subset. Hence, the set of function $f:\mathbb{N}\to\{0,1\}$ can be placed in a 1-1 correspondence with the power set $P(\mathbb{N})$ whose cardinality is the cardinality of the continuum.

2. **(Well-Ordered Sets)** A set is said to be well-ordered if every non-empty subset of the set has a least element. The usual ordering $\leq$ of the integers is not a well-ordering since, for example, the set itself has smallest element. However, the following relation $\prec$ is a well-ordering of the integers

$$x\prec y \Leftrightarrow \left[\left(|x|<|y|\right)\vee\left(|x|=|y|\wedge\left(x\leq y\right)\right)\right]$$

3. (**Well-Ordering the Integers**) Order the integers according to this well-ordering.
Ans: $0,-1,1,-2,2,-3,3,-4,4,-5,5,\ldots$

4. **(Online Research)** Google some of the words and phrases, Zermelo – Fraenkel axioms, Cantor's theorem, Zermelo, well-ordering principle, …and learn what other have to say about these topics. Readers of any math book should check the internet for more information.

ΦΣΖΛΩΙΘ