Problems 3.1: Relations

1, **True or False** Tell whether the following statements are true or false.

a) $\mathbb{Q} \times \mathbb{Q} \subseteq \mathbb{R} \times \mathbb{R}$

Ans: True

b) $aRb \Rightarrow (a,b) \in R$

Ans: True, that is the definition of a relation

c) $aRb \Rightarrow bRa$

Ans: False, for example if $a,b \in \mathbb{R}$ and $R = <$.

d) aRa

Ans: false, not always, for example if $a,b \in \mathbb{R}$ and $R = <$.

e) For any two sets A, B, $A \times B = B \times A$

Ans: False, not true when $A \neq B$.

f) For some sets A, B, $A \times B = B \times A$

Ans: True if $A = B$

2. **(Relations as a Set)** For the set $A = \{1,2,3,4\}$ write out the ordered pairs in the relation R on A if

a) $xRy \Leftrightarrow x < y$
b) $xRy \Leftrightarrow x = y$
c) $xRy \Leftrightarrow x$ divides y
d) $xRy \Leftrightarrow x$ is a multiple of y

Ans:

a) $R = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$
b) $R = \{(1,1),(2,2),(3,3),(4,4)\}$
c) $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$
d) $R = \{(1,1),(2,1),(3,1),(4,1),(2,2),(4,2),(3,3),(4,4)\}$

3. **(Four Basic Cartesian Products)** Let $A=\{1,2,3\}, B=\{a,b\}$. Find the following Cartesian products.

a) $A\times B$
Ans: $A\times B=\{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$

b) $B\times A$
Ans: $B\times A=\{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$

c) $A\times A$
Ans: $A\times A=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

d) $B\times B$
Ans: $B\times B=\{(a,a),(a,b),(b,a),(b,b)\}$

4. **(Cartesian Products)** For each pair of sets A and B find the Cartesian products $A\times B$ and $B\times A$.

a) $A=\{0,2\}$, $B=\{-1,0\}$
Ans: $A\times B=\{(0,-1),(0,0),(2,-1),(2,0)\}$
$B\times A=\{(-1,0),(-1,2),(0,0),(0,2)\}$

b) $A=\{a,b\}$, $B=\{b,c\}$
Ans: $A\times B=\{(a,b),(a,c),(b,b),(b,c)\}$
$B\times A=\{(b,a),(b,b),(c,a),(c,b)\}$

c) $A=\mathbb{R}$, $B=\mathbb{N}$
Ans: $\mathbb{R}\times\mathbb{N}=\{(x,n):x\in\mathbb{R},\ n\in\mathbb{N}\}$
$\mathbb{N}\times\mathbb{R}=\{(n,x):n\in\mathbb{N},\ x\in\mathbb{R}\}$

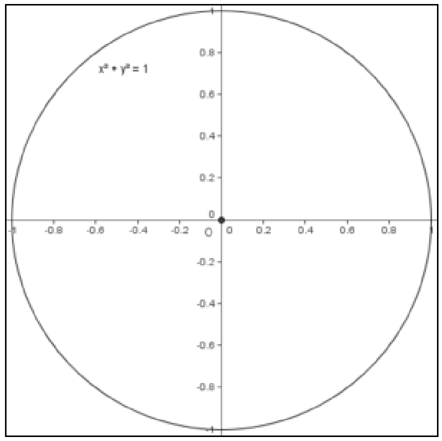
d) $A=\mathbb{Z}$, $B=\mathbb{N}$
Ans: $\mathbb{R}\times\mathbb{N}=\{(x,n):x\in\mathbb{R},\ n\in\mathbb{N}\}$
$\mathbb{N}\times\mathbb{R}=\{(n,x):n\in\mathbb{N},\ x\in\mathbb{R}\}$

e) $A=\mathbb{R}$, $B=\{-1,0,1\}$
Ans: $\mathbb{R}\times B=\{(x,-1),(x,0),(x,1):x\in\mathbb{R}\}$
$B\times\mathbb{R}=\{(-1,x),(0,x),(1,x):x\in\mathbb{R}\}$

5. **(Graphing a Relation)** Sketch the following relations.

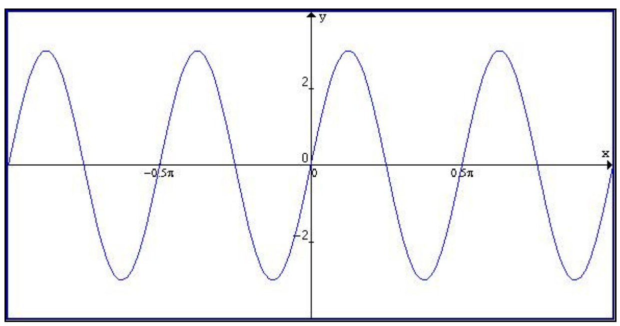
a) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$

Ans:



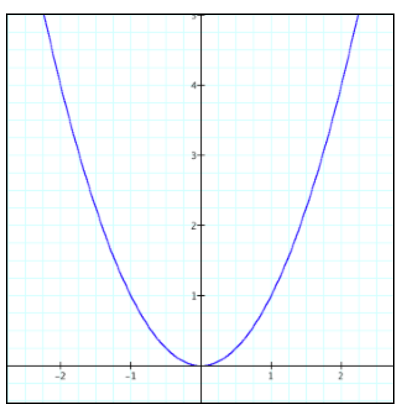
b) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \sin x\}$

Ans:

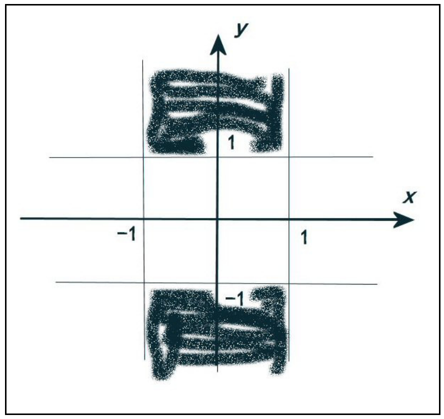


c) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y^2\}$

Ans:

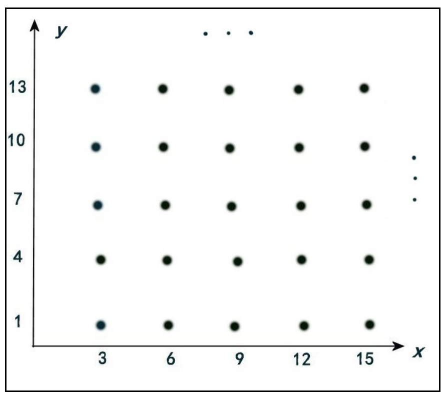


d) $R=\{(x,y)\in\mathbb{R}\times\mathbb{R}:|x|\le 1,\ |y|\ge 1\}$



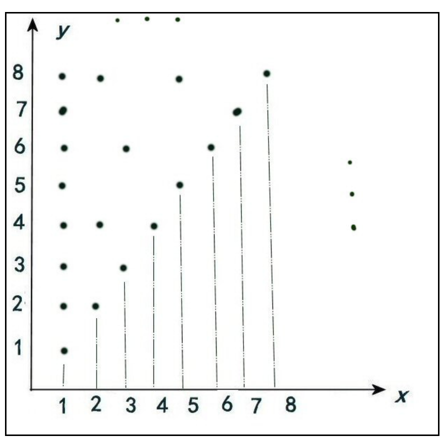
e) $R=\{(x,y)\in\mathbb{N}\times\mathbb{N}:x\equiv 0(\mathrm{mod}\,3),\ y\equiv 1(\mathrm{mod}\,3)\}$

Ans:



f) $R=\{(x,y)\in\mathbb{N}\times\mathbb{N}:x\text{ divides }y\}$

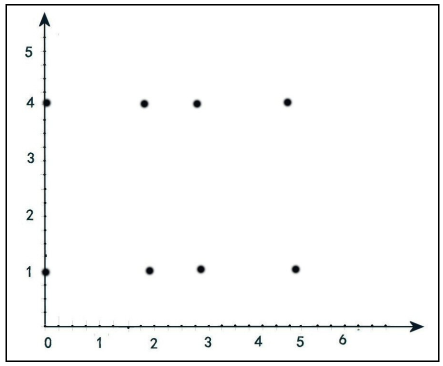
Ans:



6 (**Algebra of Relations**) Suppose $A=\{0,3\}$, $B=\{2,5\}$, $C=\{1,4\}$ are closed intervals on the real $\mathbb{R}$. Sketch the following relations $R\subseteq\mathbb{R}\times\mathbb{R}$.

a) $R=(A\cup B)\times C\subseteq\mathbb{R}\times\mathbb{R}$

Ans:

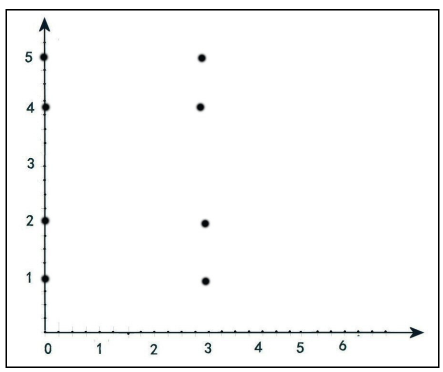


b) $R=(A\cap B)\times C\subseteq\mathbb{R}\times\mathbb{R}$

Ans: The relation is empty so there are no points in the plane.

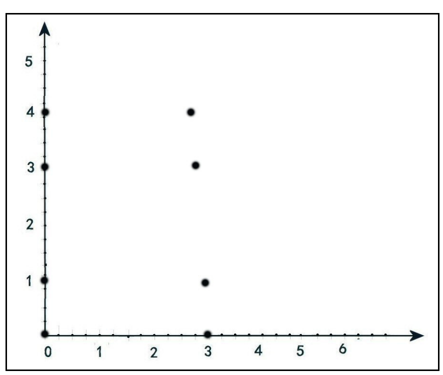
c) $R=(A\times B)\cup(A\times C)\subseteq\mathbb{R}\times\mathbb{R}$

Ans:



d) $R=A\times(A\cup C)\in\mathbb{R}\times\mathbb{R}$

Ans:



7. **(Naming a Relation)** Give a name which describes the following relations on $A = \{1,2,3\}$, then find the inverse relation of each relation. What is a name for the inverse relation?

a) $R = \{(1,1),(2,2),(3,3)\}$

Ans: $R^{-1} = \{(1,1),(2,2),(3,3)\}$, "equals" relation describes both R and R^{-1} ",

b) $R = \{(1,2),(1,3),(2,3)\}$

Ans: $R^{-1} = \{(2,1),(3,1),(3,2)\}$, "less than" describes R, "greater than" describes R^{-1}.

c) $R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

Ans: $R^{-1} = \{(1,1),(2,1),(3,1),(2,2),(3,2),(3,3)\}$, "less than or equals" describes R, "greater than or equals" describes R^{-1}.

8. **(Important Types of Relations)** Two important types of relations are injective and surjective relations, whose definitions are given below in predicate logic notation. For $X = Y = \{1,2,3\}$ give an example of an injective and surjective relation from X to Y .

a) **Surjective:** A relation $R \subseteq X \times Y$ is surjective if

$$(\forall y \in Y)(\exists x \in X)(xRy)$$

Ans: $R = \{(1,1),(2,2),(3,3)\}$

b) **Injective:** A relation $R \subseteq X \times Y$ is surjective if

$$(\forall x_1, x_2 \in X)(\forall y \in X)\left[(x_1Ry) \wedge (x_2Ry) \Rightarrow (x_1 = x_2)\right]$$

Ans: $R = \{(1,1),(2,2),(3,3)\}$ The identity relation is an injective relation.

9 **(Hmmmmmmmmmmm)** Consider the relation $R \in \mathbb{N} \times \mathbb{N}$ defined by

$R = \{(m,n) : m \le n\}$. Find the following compositions.

a) $R \circ R^{-1}$
b) $R \circ R$

Ans: Student Project

10**.** **(Meaning of Relations)** Give examples of elements related for the following relations R.

a) $R \in \mathbb{Q} \times \mathbb{Q},\ R = \left\{ \left(p/q, r/s \right) : ps = qr \right\}$

Ans: Pairs of fractions that are equal, like $\frac{1}{2} = \frac{3}{6} = \frac{-5}{10} = \cdots$.

b) $R \in \mathbb{Q} \times \mathbb{Q},\ R = \varnothing$, $mRn \Leftrightarrow m - n$ is irrational

Ans: No two rational numbers are related.

c) $R \in \mathbb{R} \times \mathbb{R},\ R = \mathbb{R}$

Ans: All real numbers are related.

11. **(Inverse Relation of Compositions)** Given the set $A = \{1, 2\}$ verify the identity for the inverse of a composition $\left(S \circ R \right)^{-1} = R^{-1} \circ S^{-1}$ for relations

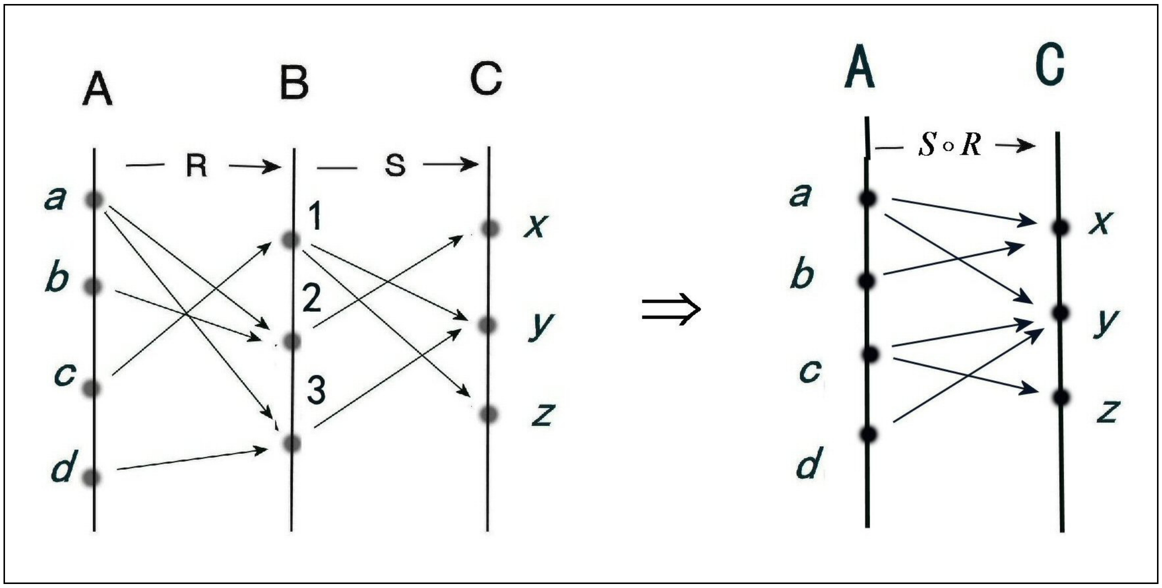
$$R = \left\{ (1,2), (2,3) \right\} \subseteq A \times A$$
$$S = \left\{ (2,2), (3,1) \right\} \subseteq A \times A$$

Ans: $S \circ R = \left\{ (1,2), (2,1) \right\} \Rightarrow \left(S \circ R \right)^{-1} = \left\{ (2,1), (1,2) \right\}$,
$R^{-1} = \left\{ (2,1), (3,2) \right\},\ S^{-1} = \left\{ (2,2), (1,3) \right\} \Rightarrow R^{-1} \circ S^{-1} = \left\{ (2,1), (1,2) \right\}$

12. **(Composition of Relations)** Given $A = \{a, b, c, d\}, B = \{1, 2, 3\}, C = \{x, y, z\}$ and

$$R = \left\{ (a,2), (a,3), (b,2), (c,1), (d,3) \right\} \subseteq A \times B$$
$$S = \left\{ (2,x), (1,y), (1,z), (3,y) \right\} \subseteq B \times C$$

Ans: $S \circ R = \left\{ (a,x), (a,y), (b,x), (c,y), (c,z), (d,y) \right\} \subseteq A \times C$



13. **(Cartesian Product Identities)** Prove $(A-B)\times B=(A\times B)-(B\times B)$.

Ans:

$(\subseteq)$ Let $x\in(A-B)\times B$ and so there exists $p\in A-B$ and $q\in B$ such that $x=(p,q)$. Now, since $p\in A,\ q\in B$ we have $(p,q)\in A\times B$. We also know that $p\notin B$ and so $(p,q)\notin B\times B$. Hence, we have $x=(p,q)\in(A\times B)-(B\times B)$ which proves $(A-B)\times B\subseteq(A\times B)-(B\times B)$.

$(\supseteq)$ Let $x\in(A\times B)-(B\times B)$. Therefore there exists a $p\in A,\ q\in B$ such that $x=(p,q)$. Since $x\notin B\times B$ and $q\in B$, then $p\notin B$. Thus $p\in A-B$ and $x=(p,q)\in(A-B)\times B$. Hence $(A\times B)-(B\times B)\subseteq(A-B)\times B$ which proves the result.

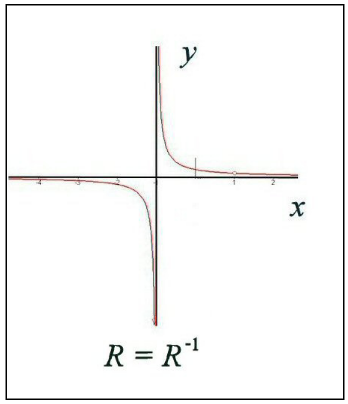
14. **(Number of Relations)** If a set *A* has m elements and *B* has n elements then show there are 2^{mn} distinct relations from *A* to *B*.

Ans: The Cartesian product $A\times B$ has mn elements and since each relation from A to B is is a subset of $A\times B$ of which there are are 2^{mn} subsets, we have the desired result.

15. **(Graphing Relations and Their Inverses)** Graph the following relations and their inverses.

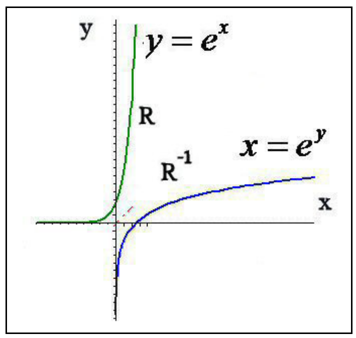
a) $R\subseteq\mathbb{R}\times\mathbb{R},\ xRy\Leftrightarrow y=1/x$

Ans:



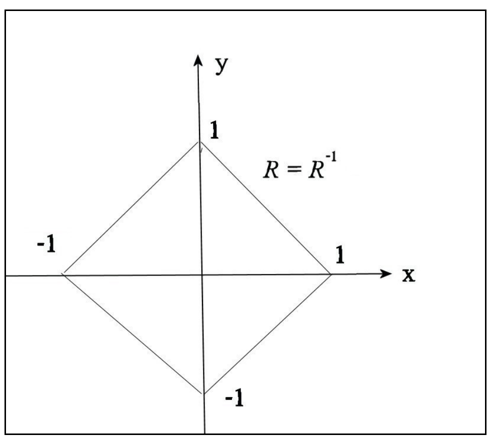
b) $R \subseteq \mathbb{R}\times\mathbb{R},\ xRy \Leftrightarrow y = e^x$

Ans:



c) $R \subseteq \mathbb{R}\times\mathbb{R},\ R = \{(x, y) : |x| + |y| = 1\}$

Ans:



16. **(Counting Relations I)** Given the set $A = \{1,2\}$:, how many relations are thene on A ?

Ans:

$A \times A$ has 4 elements and this set has $2^4 = 16$ subsets. So the answer is 16. Try writing them down.

17. **(Counting Relations II)** Given a set $A = \{1,2,3\}$, how many relations are there on A?
Ans:

Number of relations: The Cartesian product $A \times A$ contains 9 members and we are free to include or not include in the relation any of these $2^9 = 512$ possibilities. For a set with n members there would be 2^{2^n} relations.

18. **(Irreflexive Relation)** A relation on a set A is **irreflexive** if $(\sim \exists x \in A)(xRx)$, or equivalently $(\forall x \in A)(\sim xRx)$. In other words, no $x \in A$ is related to itself. Find as many irreflexive relations as you can. Choose the set A anyway you wish.
Ans: Here are a few:

- less than on $\mathbb{R}$ since no number is less than itself greater than on $\mathbb{R}$
- is a sibling of" since no one can be a sibling to themselves
- is a proper subset"
- not equal to"
- is bigger than"

19**. (Converse of a Binary Relation)** If R is a binary relation on a set A, then the converse relation $\breve{R}$ on A is defined by $x\breve{R}y \Leftrightarrow yRx$. State in English or write out in ordered pairs the converse of the following relations R. In some cases the set A on which the relation is defined is specified.

a) The relation "is the mother of."
b) The relation "is a uncle of."
c) The relation "<" on the real numbers.
d) The relation "=" on the real numbers.

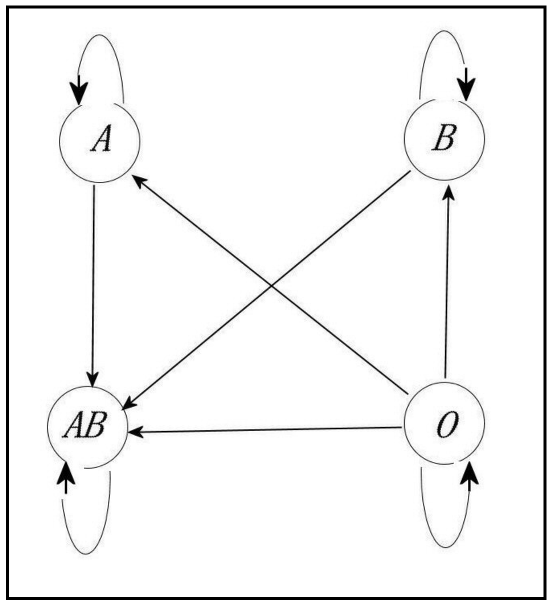
Ans: a) "is the son or daughter of"
a) is the nephew or neice of"
b) ">"
c) "="

20**. (Blood Typing)** There are four types of blood, types $A, B, AB,$ and O. A person with type A can receive blood from a person with types A and O; a person with type B can receive blood from a person with types B and O; a person with type AB can receive blood from persons of all types; and a person of type O can receive blood from persons only of type O. Given the set $S=\{A,B,AB,O\}$ define the binary relation R on S as follows: xRy if and only if a person of type x can receive blood from a person of type y.

a) Write out the ordered pairs of R.
b) Draw a directed graph of the relation R.
c) Defining the converse $\breve{R}$ of R by $x\breve{R}y \Leftrightarrow yRx$ what is the interpretation of the converse?
d)

Ans: a) $R=\{(A,A),(A,AB),(B,B),(B,AB),(AB,AB),(O,A),(O,B),(O,AB),(O,O)\}$

b)



c) $x\breve{R}y$ iff a person of type x can give blood to a person of type y.

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