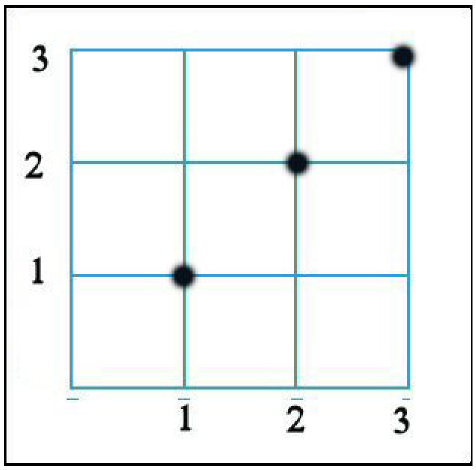
Section 3.2 Order Relations

1. **(Testing for an Order Relation)** Tell whether the following relations on $A=\{1,2,3\}$ are reflexive, antisymmetric, and transitive. Plot the points of the Cartesian product $A\times A$ and denote the members of $R\subseteq A\times A$. If the relation is an order relation, draw a Hasse diagram and directed graph.

a) $R=\{(1,1),(2,2),(3,3)\}$

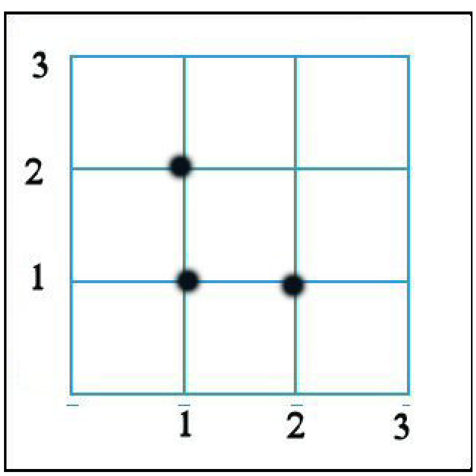
Ans: reflexive, antisymmetric and transitive



b) $R=\{(1,1),(1,2),(2,1)\}$

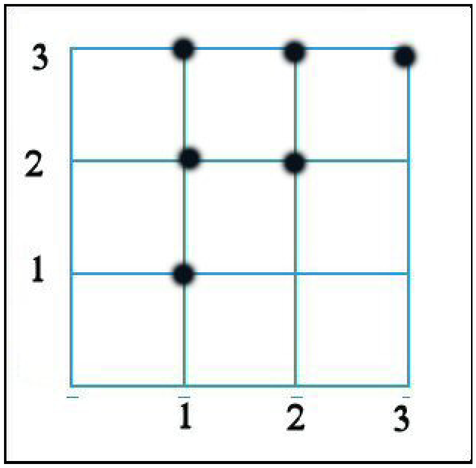
Ans: not reflexive since $2\not R\,2$

not antisymmetric since $2R1 \wedge 1R2$ but $2\not R\,2$ is transitive



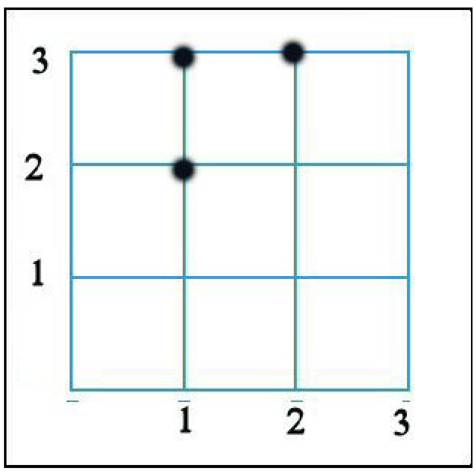
c) $R=\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\}$

Ans: reflexive, antisymmetric, transitive



d) $R = \{(1,2),(2,3),(1,3)\}$

Ans: not reflexive, antisymmetric, transitive



1. **(Finding Relations)** Find a relation on the $A = \{1,2,3,4\}$ with the following properties.

a) reflexive but not antisymmetric

Ans: $R \subseteq A \times A,\ R = \{(1,1),(2,2),(3,3),(4,4)\}$

a) antisymmetric and reflexive

Ans: $R \subseteq A \times A,\ R = \{(1,1),(2,2),(3,3),(4,4)\}$ A more interesting relation might be

$$R = \{(1,1),(1,2),(1,4),(1,4),(2,2),(2,3),(2,4),(3,3),(3.4),(4,4)\}$$

which is the relation $x \le y$

b) not reflexive but transitive

Ans: $R \subseteq A \times A,\ R = \{(1,1),(2,2),(3,3)\}$

c) not reflexive, not antisymmetric, not transitive
Ans: $R = \{(1,2),(2,4),(2,1)\}$

2. **(Ordering of Functions)** Let $C[0,1]$ be the set of continuous functions defined on $[0,1]$. For
$f, g \in C[0,1]$, define the ordering

$$f \le g \Leftrightarrow (\forall x \in [0,1])[f(x) \le g(x)].$$

Show that "$\le$" defines a partial order on $C[0,1]$.
Ans: Reflexive: $f \le f$ since $f(x) \le f(x)$ for all $x \in [0,1]$
Antisymmetric: If $f \preceq g$ then $f(x) \le g(x)$ for all $x \in [0,1]$, and if $g \le f$ then $g(x) \le f(x)$ for all $x \in [0,1]$. But this implies $f(x) = g(x)$ for all $x \in [0,1]$, or $f = g$.

Transitive: This is straightforward and left to the reader.

3. **(Upper and Lower Bounds)** For the partially ordered set $P(A)$ with order relation $\subseteq$ in Example 6, find an upper bound, least upper bound, a lower bound, and the greatest lower bound for the following subsets of $P(A)$.

a) $B = \{\{a\},\{a,b\}\}$
Ans:

$$\text{UB} = \{\{a,b\},\{a,b,c\}\}$$
$$\text{LUB} = \{a,b\}$$
$$\text{LB} = \{\varnothing,\{a\}\}$$
$$\text{GLB} = \{a\}$$

b) $B = \{\{a\},\{b\}\}$
Ans:

$$\text{UB} = \{\{a,b\},\{a,b,c\}\}$$
$$\text{LUB} = \{a,b\}$$
$$\text{LB} = \varnothing$$
$$\text{GLB} = \varnothing$$

c) $B = \{\{a\},\{a,b\}\{a,b,c\}\}$
Ans:

$\text{UB} = \{a,b,c\}$
$\text{LUB} = \{a,b,c\}$
$\text{LB} = \{\varnothing,\{a\}\}$
$\text{GLB} = \{a\}$

d) $B = \{\{a\},\{c\},\{a,c\}\}$
Ans:
$\text{UB} = \{\{a,c\},\{a,b,c\}\}$
$\text{LUB} = \{a,c\}$
$\text{LB} = \varnothing$
$\text{GLB} = \varnothing$

e) $B = \{\varnothing,\{a,b,c\}\}$
Ans:
$\text{UB} = \{a,b,c\}$
$\text{LUB} = \{a,b,c\}$
$\text{LB} = \varnothing$
$\text{GLB} = \varnothing$

f) $B = \{\{a\},\{b\},\{c\}\}$
Ans:
$\text{UB} = \{a,b,c\}$
$\text{LUB} = \{a,b,c\}$
$\text{LB} = \varnothing$
$\text{GLB} = \varnothing$

4. **(Sups and Infs)** Find the sup, inf, max, min (if they exist) of the following sets.
a) $(-\infty,2)$
b) $(-\infty,2]$
c) $\left\{1+\frac{1}{n} : n \in \mathbb{N}\right\}$
d) $\left\{\frac{1}{m}+\frac{1}{n} : m,n \in \mathbb{N}\right\}$
e) $\left\{\frac{1}{m}-\frac{1}{n} : m,n \in \mathbb{N}\right\}$
f) $\left\{\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},\ldots\right\}$ (We omit fractions not in reduced form.)

g) $\left\{ \frac{n}{n^2+1} : n \in \mathbb{N} \right\}$

h) $\left\{ y : y = x^2 + x - 2,\ x \in \mathbb{R} \right\}$

Ans:

a) for the set $(-\infty, 2)$, no min, no inf, no max, sup = 2

b) for the set $(-\infty, 2]$, no min, no inf, max = 2, sup = 2

c) for the set $\left\{ 1 + \frac{1}{n} : n \in \mathbb{N} \right\}$, no min, inf = 1, max = sup = 2

d) for the set $\left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \right\}$, max = sup = 2, no min, inf = 0

e) for the set $\left\{ \frac{1}{m} - \frac{1}{n} : m, n \in \mathbb{N} \right\}$, no max, sup = 1, no min, inf = -1

f) for the set $\left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots \right\}$, max = sup = ½, no min, inf = 0

g) for the set $\left\{ \frac{n}{n^2+1} : n \in \mathbb{N} \right\}$, max = sup = ½, no min, inf = 0

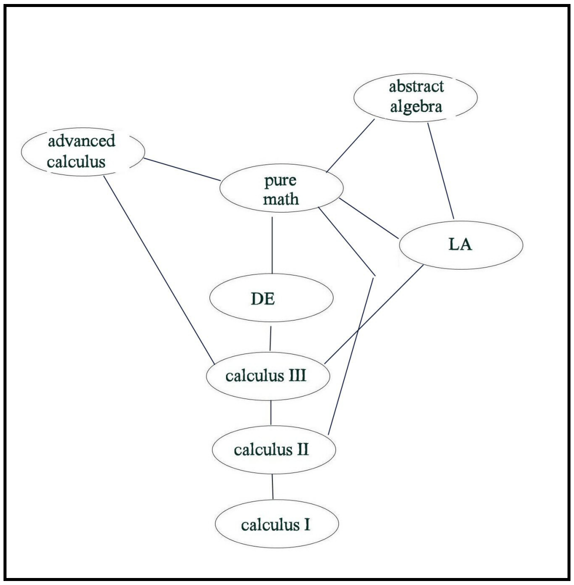
h) for the set $\left\{ y : y = x^2 + x - 2,\ x \in \mathbb{R} \right\}$, min = inf = -9/4, no max, no sup.

5. **(Hasse Diagram)** Jane is getting a degree in mathematics and has several courses to take. Some of the courses have prerequisites as shown below.

Course Needed	**Prerequisites**
Calculus I	
Calculus II	Calculus I
Calculus III	Calculus II
Linear Algebra	Calculus III
Differential Equations	Calculus III
Intro to Pure Math	Linear Algebra, Calculus II
Abstract Algebra	Linear Algebra, pure math
Advanced Calculus	Calculus III, pure math

Draw a Hasse diagram that illustrates the order which Jane must take the courses.

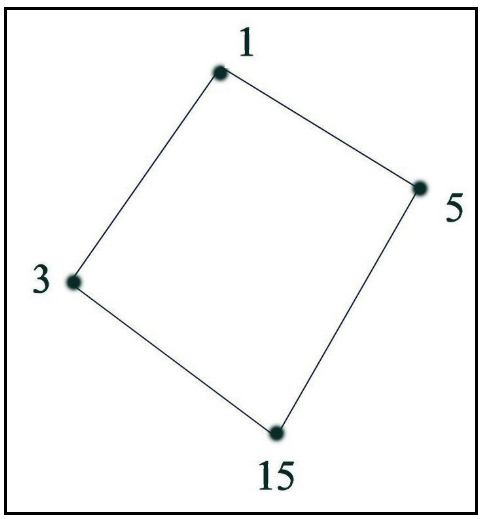
Ans:



6. **(Hasse Diagram for Multiples)** Let M be the order relation "a is a multiple of b" defined on the set of positive divisors of 15. Draw a Hasse diagram for M .

Ans:

:



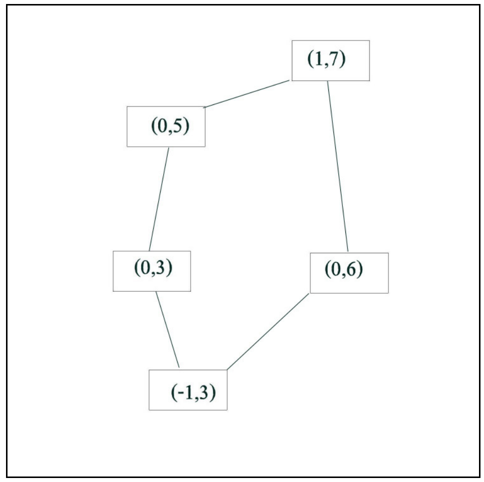
7. **(A Partial Order for Points in the Plane)** There are various ways to construct new orders from existing orders. A partial order can be constructed on the Cartesian product of two partially ordered sets by defining

$$(a,x) \le (b,y) \Leftrightarrow (a \le b) \wedge (x \le y)$$

a) Construct a Hasse diagram that represents a partial order for

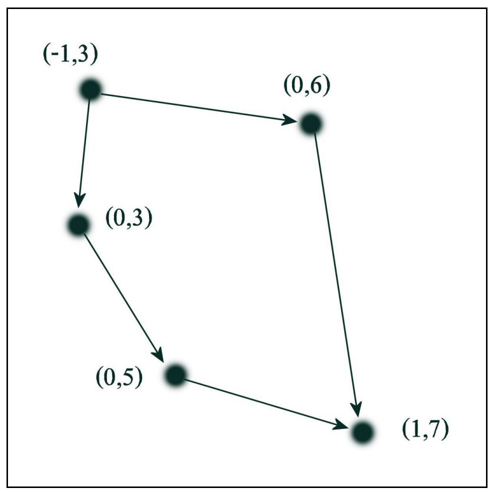
$$A = \{(-1,3),(0,3),(1,7),(0,6),(0,5)\}.$$

Ans:



b) Draw a digraph for the set A.

Ans:



8. **(Equivalent form of Antisymmetry)** State the contrapositive form of the anti-symmetry condition

$$(x \le y) \wedge (y \le x) \Rightarrow x = y.$$

Ans: $x \ne y \Rightarrow (x > y) \vee (y > x)$

9. **(Ordering the Complex Numbers)** Suppose you order the complex numbers $z = a + bi$ according to their magnitude $|z| = \sqrt{a^2 + b^2}$, that is

$$z_1 \le z_2 \Leftrightarrow |z_1| \le |z_2|$$

Is this a partial order ?

Ans: It is not a partial order since the relation is not antisymmetric. For example $|1| \le |i|$ and $|i| \le |1|$ but $1 \ne i$. However the properties of reflexive and transitive hold.

10. **(Total Order of the Complex Numbers)** The complex numbers can be totally ordered as follows. Given two complex numbers in polar form $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, $r_1, r_2 \geq 0,\ 0 \leq \theta < 2\pi$ we define the order

$$z_1 \leq z_2 \Leftrightarrow \begin{cases} r_1 < r_2 \\ r_1 = r_2,\ \theta_1 < \theta_2 \end{cases}$$

Compare the following complex numbers.

a) $z_1 = i,\ z_2 = 1 + i$ **Ans**: $z_1 \preceq z_2$ since $r_1 = 1 \leq r_2 = \sqrt{2}$

b) $z_1 = i,\ z_2 = -1$ **Ans**: $z_1 \preceq z_2$ since $r_1 = r_2 = 1,\ \theta_1 = \pi / 2 \leq \theta_2 = \pi$

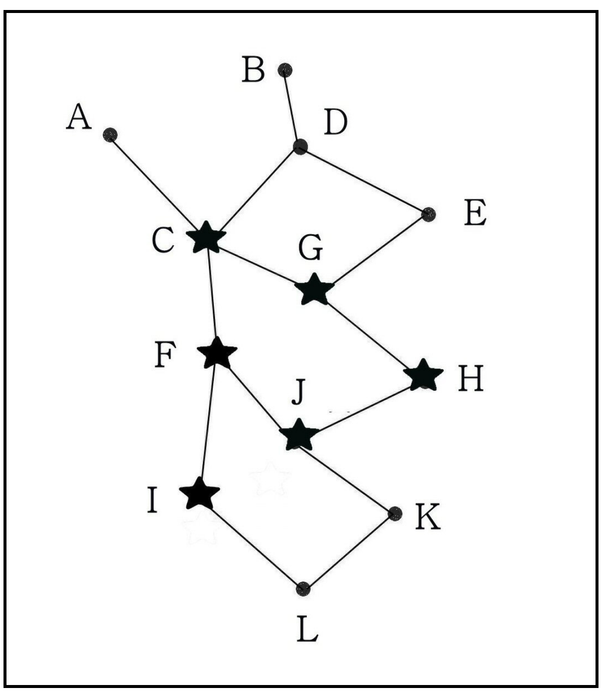
c) $z_1 = 6i,\ z_2 = 2 + 3i$ **Ans**: $z_2 \preceq z_1$ since $r_1 = 6 \geq r_2 = 5$

d) $z_1 = 0,\ z_2 = 1$ **Ans**: $z_1 \preceq z_2$ since $r_1 = 0 \leq r_2 = 1$

11. **(Counting Partial Orders)** There are a total of three partial orders on the set $A = \{1,2\}$. Can you find them?

Ans: $R_1 = \{(1,1),(2,2)\}$, $R_2 = \{(1,1),(2,2),(1,2)\}$, $R_1 = \{(1,1),(2,2),(2,1)\}$ .In other words $=, \leq$ and $\geq$. If you want a more challenging problem find the 19 partial orders on a set of 3 elements, or the 219 partial orders on a set of 4 elements.

12. **(Hasse Diagram)** For the "starred" subset $S = \{C, F, G, I, J, H\}$ of the partially ordered set $U = \{A, B, C, D, E, F, G, H, I, J, K, L\}$ illustrated below, find the following:



a)	upper bound(s)	**Ans:** A,B,C,D
b)	lower bound(s)	**Ans:** L
c)	the least upper bound	**Ans:** C
d)	greatest lower bound	**Ans:** L
e)	maximal element(s)	**Ans:** A,B
f)	minimal element(s)	**Ans:** L
g)	maximum	**Ans:** C
h)	minimum	**Ans:** does not exist

13. **(Test Your Knowledge of Sups and Infs)** A Hasse diagram representation of a partial order is shown in the diagram below. Find, if they exist, the sup and inf of the following sets. A partially ordered set is called a **lattice** if every every pair of elements in the set has a sup and inf. Is the given set under this partial order a lattice?

a) $\{2,3\}$

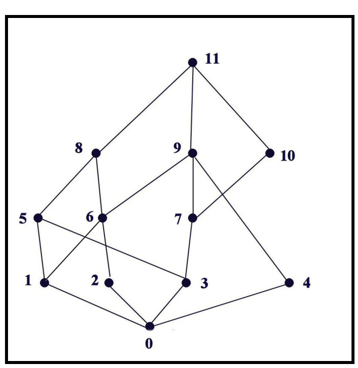
Ans: $\sup\{2,3\}$ does not exist since the upper bounds for $\{2,3\}$ are $\{8,9,11\}$ which has no least member. $\inf\{2,3\}=0$ since the lower bounds of $\{2,3\}$ are $\{0\}$. Hence, the given set is not a lattice since we have found one pair of elements that does not have a supremum or infimum.

b) $\{8,9\}$

Ans: $\sup\{8,9\}=11$ since 11 is the only upper bound of both 8 and 9. Also, $\inf\{8,9\}=6$ since the lower bounds of $\{8,9\}$ are $\{0,2,6\}$ and 6 is the greatest of these.

c) $\{1,2\}$

Ans: $\sup\{1,2\}=6$ since the upper bounds of $\{1,2\}$ is $\{6,8,9,11\}$, $\inf\{1,2\}=0$ since the lower bounds of $\{1,2\}$ are $\{0\}$..



14. **(SUP or MAX?)** Given the set

$$S=\left\{\frac{n}{n+1}:n=1,2,3,\ldots\right\}$$

find the maximum of the set if it exists. If it does not exist, find an upper bound and the least upper bound of S.

Ans: Writing out a few members of the set, we see

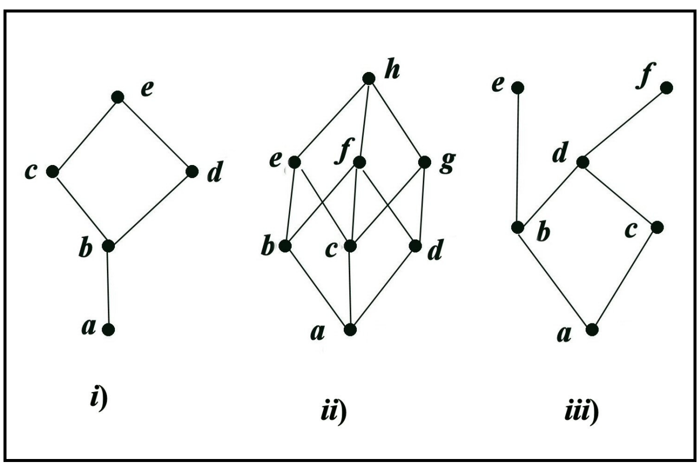
$$S=\left\{\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\ldots\right\}$$

Clearly, none of the members in the set is the maximum value of the set. However, any number greater than or equal to 1 is an upper bound. We claim that 1 is the least upper bound since if there *is* a smaller upper bound, say $1-\varepsilon$ where $\varepsilon>0$, then for some $n\in\mathbb{N}$ sufficiently large, we can make

$$1-\varepsilon<\frac{n}{n+1}\ .$$

Hence, $1-\varepsilon$ is not an upper bound which contradicts our claim $1-\varepsilon$ was an upper bound. Hence, we must conclude that 1 is the least upper bound (or SUP) for S

15. **(Lattices)** A lattice L is a partially ordered set in which every two members $a,b\in L$ has a supremum (sup) and infimum (inf) in L.The supremum is often called the **join** of a and b and denoted by $a\vee b$, and the infimum is called the **meet** of a and b and denoted by $a\wedge b$. Determine which of the following partially ordered sets, represented by Hasse diagrams, are lattices.



Ans: i) is a lattice; for example

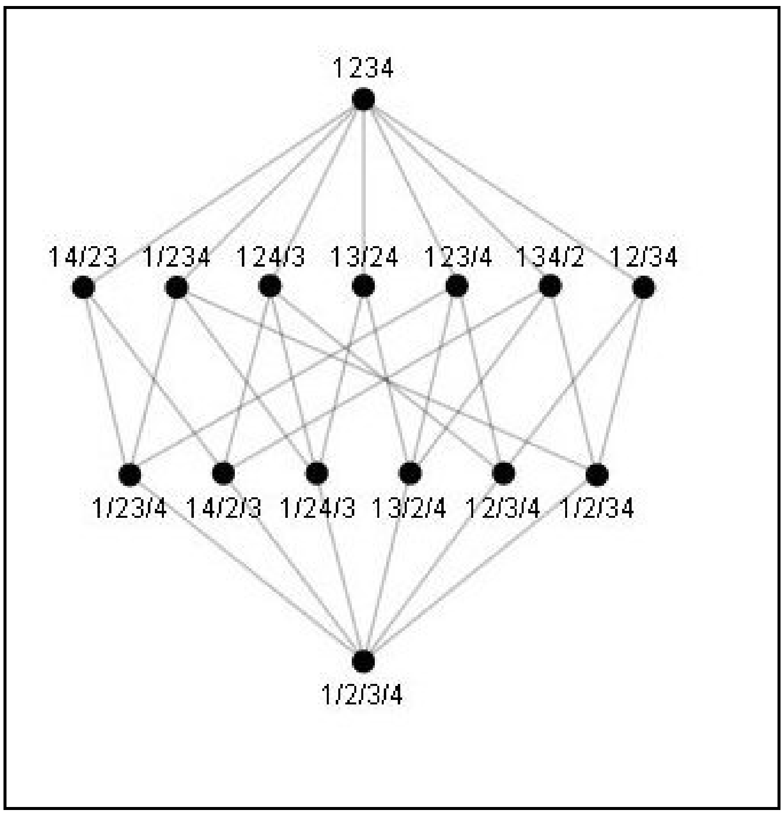
$$\sup\{c,d\}=c\vee d=e,\ \inf\{c,d\}=c\wedge d=b$$
$$\sup\{a,b\}=a\vee b=b,\ \inf\{a,b\}=a\wedge b=a$$

You can work out the other sups and infs.

Ans: ii) is not a lattice; for example $\sup\{c,d\}$ does not exist since the upper bounds for $\{c,d\}$ are $\{f,g,h\}$ but this set has no smallest member.

Ans: iii) is not a lattice; for example $\sup\{e,f\}$ does not exist since the upper bounds for $\{e,f\}$ is empty.

16 **(Lattice of Partitions)** A **lattice** is a partially ordered set in which every two elements has a unique least upper bound, called their **join**, and a unique greatest lower bound, called their **meet** . Figure 10 shows a Hasse diagram for the set of all partitions of $\{1,2,3,4\}$ into disjoint subsets, partially ordered by "increasing merging of sets." The slashes between numbers represents different partitions. For instance $1/2/3/4$ means the partition $\{1\},\{2\},\{3\},\{4\}$ and $14/23$ denotes the partition $\{1,4\},\{2,3\}$. Draw the lattice for the set of partitions of the set $\{a,b,c\}$.



17. **(True or False)** Answer the following true or false. For rational and real numbers we assume the usual partial ordering "$\le$".

a) Every partially ordered set has a least upper bound.
b) Every set that is bounded above has a least upper bound.

c) Every set of rational numbers that is bounded above has a least upper bound.
d) Every set of real numbers that is bounded above has a least upper bound.

Ans:

a) No, the real numbers has no least upper bound since they are not bounded above.
b) No, let S be the set of positive rational numbers that are less than $\sqrt{2}$. That is

$$S = \left\{ 0 < r < \sqrt{2} \;:\; r \in \mathbb{Q} \right\}$$

Clearly, $\sqrt{2}$ is an upper bound for S, but $\sqrt{2} = 1.41421356\ldots \notin S$ is an irrational number and the limit of rational numbers $1.4, 1.41, 1.414, \ldots$ etc, all of which are less than $\sqrt{2}$ and belong to S. Hence, $\sup(S) = \sqrt{2} \notin S$ which means that S is bounded above but does not have a least upper bound.

c) No, the answer is the same as in part b)
d) The answer is yes, but we must wait until Chapter 5 to see why.

18. **(Dense Orders)** A partial order R on a set A is said to be dense in A if

$$(\forall x, y \in A)\left[xRy \Rightarrow (\exists z \in A)(xRz \wedge zRy) \right]$$

Which of the following partial orders are dense in the given set?

a) The "less than or equal to" order $\le$ on the rational numbers $\mathbb{Q}$.
b) The "containment" relation $\subseteq$ on the power set $P(A)$ of a set.
c) The "less than" relation < on the real numbers $\mathbb{R}$.
d) The relation "is younger than" on a collection of people.

Ans: a) Yes, if $p \le q$ for rational numbers p, q there is always a rational number r such that $p \le r \le q$. In case $p = q$ one simply selects $r = p = q$.

a) No, in case $P(\{a,b,c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ we have $\{a\} \subseteq \{a,b\}$
but there is no member $X \in P(\{a,b,c\})$ such that $\{a\} \subseteq X \subseteq \{a,b\}$

b) Yes, if $x < y$ for real numbers x, y, is always a real number z such that $x < z < y$.

c) No, for example if Bob is younger than Jane, that doesn't mean there is someone in the group
whose age is between that of Bob and Jane.

19. **(Inverse of a Partial Order)** If R is a partial order on a set A, then the inverse R^{-1} relation is also a partial order on A.

Ans: Since R is a partial order it satisfies

- $(x,x) \in R$ for all $x \in A$ reflexive
- $[(x,y) \in R \wedge (y,x) \in R] \Rightarrow x = y$ antisymmetric
- $[(x,y) \in R \wedge (y,z) \in R] \Rightarrow (x,z) \in R$ transitive

Since $(x,y) \in R \Leftrightarrow (y,x) \in R^{-1}$ we interchange x and y above, getting the desired relations

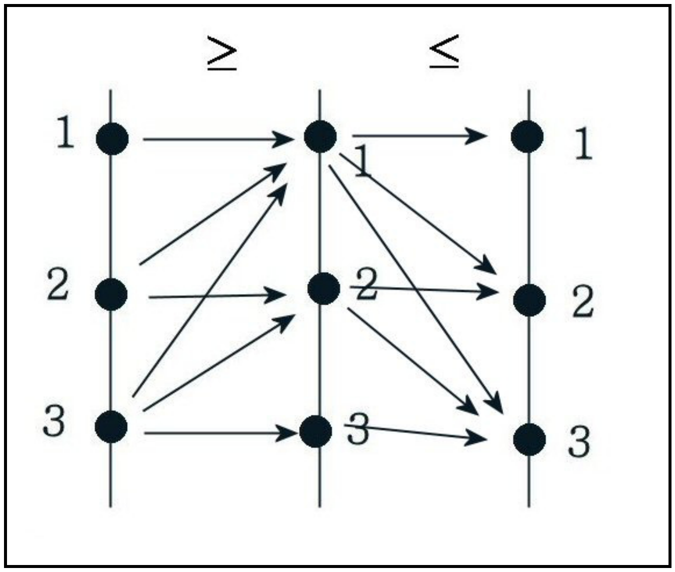
- $(x,x) \in R^{-1}$ for all $x \in A$ reflexive
- $[(y,x) \in R^{-1} \wedge (x,y) \in R^{-1}] \Rightarrow x = y$ antisymmetric
- $[(z,y) \in R^{-1} \wedge (y,x) \in R^{-1}] \Rightarrow (z,x) \in R^{-1}$ transitive

Note that the inverse relation $\leq$ on the real numbers is $\geq$.

20. (**An Upper Bounded Set with No Sup**) Show that the set
$$S = \left\{ x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2} \right\}$$
is bounded above but has no least upper bound.

Ans: Clearly the set is bounded above, for instance 5 is an upper bound. However, for any rational upper bound M; i.e. any rational number M satisfying $\sqrt{2} \leq M$ there is another rational number between $\sqrt{2}$ and M which means M is not an upper bound for S. Hence, S has no least upper bound in the set of rational numbers.

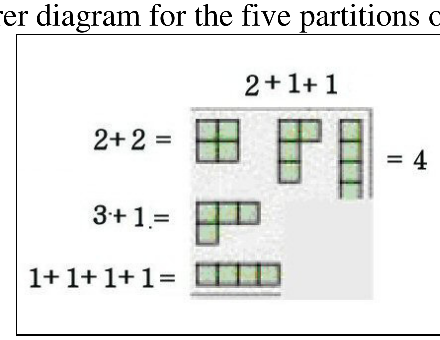
21**. (Composition of Partial Orders)** Let $R = \leq$ and $S = \geq$ be the usual "less than or equal to" and "greater than or equal to" partial orders on the set $A = \{1,2,3\}$. Find the composition $R \circ S = \leq \circ \geq$ of the two orders.
Ans: From the diagram below we see that the composition $\leq \circ \geq$ is $\{1,2,3\} \times \{1,2,3\}$. (Normally, one is not interested in compositions of partial orders.) We will see later it is functions one is generally interested in computing compositions.



22. **(Partitions of a Natural Number)**[1] A partition of a positive integer is a way of writing the number as a sum of positive integers where the order is not important but numbers can be repeated. The 5 partitions of the number 4 are

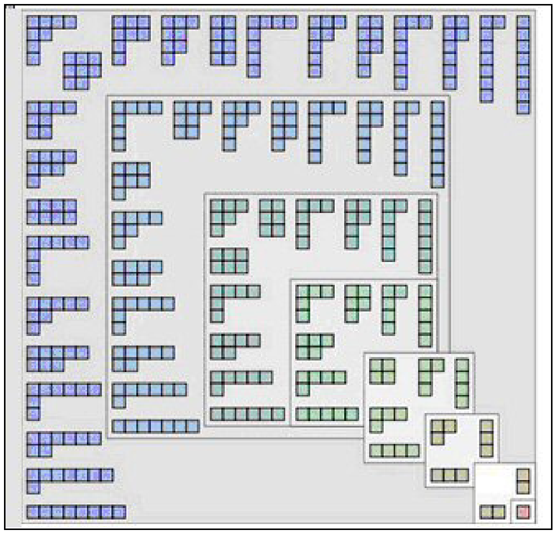
$$\begin{aligned} 4 &= 1+1+1+1 \\ &= 2+1+1 \\ &= 2+2 \\ &= 3+1 \\ &= 4 \end{aligned}$$

which we denote by the **partition function** $p(4)=5$. Although not a strict order relation, a convenient way of visually organizing partitions of small integers is by a **Ferrer diagram**. The Ferrer diagram for the five partitions of 5 is shown below.



[1] The study of integer partitions arises in combinatorial problems in unexpected ways. The subject got its big start with Euler in the 18^{th} century and today is an active area of research among additive number theorists. There are still many unsolved problems including whether (asymptotically) half the values of $p(n)$ are even and half are odd.

A Ferrer diagram illustrating the partitions of numbers 1-8 is shown in the cumulative diagram below.



a) Find the partitions of the numbers 1-6.

b) What is the value of the partition function $p(n)$, $n = 1, 2, \ldots, 8$.

c) An asymptotic estimate for $p(n)$ was found by G.H. Hardy and Ramanujan in 1918 as

$$p(n) \sim \frac{\exp\left(\pi\sqrt{2n/3}\right)}{4n\sqrt{3}} \quad \text{as } n \to \infty$$

Use this formula to estimate $p(100)$, $p(200)$, $p(1000)$.

Ans:

a) We leave this for the reader.

b) $p(1)=1, p(2)=2, p(3)=3, p(4)=5, p(5)=7, p(6)=11, p(7)=15, p(8)=22$

c) $p(25) \doteq 2145,\ p(100) \doteq 2\times10^8,\ p(200) \doteq 4\times10^{12}$

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