Section 3.3 Equivalence Relations

1. Let A be the set of students at a university and let x and y be students; i.e. elements of A. Tell if the following are equivalence relations on A.

a) x is related to y iff x and y have the same major.
Ans: It is an equivalence relation

b) x is related to y iff x and y have the GPA.
Ans: It is an equivalence relation

c) x is related to y iff x and y are from the same country.
Ans: It is an equivalence relation

d) x is related to y iff x and y have the same major.
Ans: It is an equivalence relation

2. **(Equivalence Relations?)** Tell if the following relations R are equivalence relations on a set A.
If the relation is an equivalence relation, find the equivalence classes.

a) xRy iff $y = x^2$ $(A = \mathbb{R})$

Ans: Not reflexive since $x \neq x^2$ for all real numbers x. Hence, the relation is not an equivalence relation.

b) mRn iff m is a factor of n $(A = \mathbb{N})$

Ans: The relation is not symmetric. For example, if $m = 2, n = 4$ then 2 is a factor of 4 but 4 is not a factor of 2.

c) xRy iff x and y have the same remainder when divided by 5. $(A = \mathbb{N})$

Ans: The relation is an equivalence relation. The equivalence classes are the five residue classes:

$$[0] = \{5n : n \in \mathbb{N}\}$$
$$[1] = \{1 + 5n : n \in \mathbb{N}\}$$
$$[2] = \{2 + 5n : n \in \mathbb{N}\}$$
$$[3] = \{3 + 5n : n \in \mathbb{N}\}$$
$$[4] = \{4 + 5n : n \in \mathbb{N}\}$$

d) xRy if and only if $|x-y| \le 1$, $A=\mathbb{R}$)

Ans: The relation is reflexive, symmetric, but not transitive.

e) $(a,b)R(c,d)$ if and only if $a^2+b^2=c^2+d^2$, $A=\mathbb{R}^2$.

Ans: The relation is an equivalence relation and the equivalence classes are circles centered at the origin.

3. **(Not Equivalence Relations)** The following relations on the given sets are not equivalence relation. Tell if the conditions are reflexive, symmetric, or transitive.

a) The relation "$\le$" on the real numbers.
Ans: reflexive and transitive but not symmetric

b) The empty relation on an empty set (i.e. xRy is never true)
Ans: is vacuously symmetric and transitive but not reflexive

c) The relation "$\subset$" of being a proper subset on a family of sets
Ans: transitive but not reflexive or symmetric

d) The relation of being perpendicular on the set of lines in the plane.
Ans: symmetric, but not reflexive or transitive

4. **(Finding the Equivalence Relation)** The set $A=\{a,b,c,d,e\}$ is partitioned into the equivalence classes $\{\{a,c\},\{b,e\},\{d\}\}$ by an equivalence relation $R \subseteq A \times A$. Find the relation $R \subseteq A \times A$.
Ans: $R=\{(a,a),(a,c),(c,a),(c,c),(b,b),(b,e),(e,b),(e,e),(d,d)\}$

5. **(Finding the Quotient Set)** The set

$$R=\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,1)\}$$

Is an equivalence relation on the set $A=\{1,2,3,4,5\}$. How does this relation partition A?
Ans: $A/R=\{\{1,2\},\{3\},\{4\},\{5\}\}$

6. **(Finding Equivalence Classes)** The set $A=\{1,2,3,4\}$ is partitioned into $A/R=\{\{1,2\},\{3,4\}\}$ by an equivalence relation R. Find

a) $[1]$ **Ans:** $[1]=\{1,2\}$
b) $[2]$ **Ans:** $[2]=\{1,2\}$
c) $[3]$ **Ans:** $[3]=\{3,4\}$

d) $[4]$ **Ans:** $[4]=\{3,4\}$

7. (Hmmmmmmmmm) If an equivalence relation R on a set A has only one equivalence class, what is the relation?
Ans: $R = A \times A = \{(x,y): x \in A, y \in A\}$

8. **(Unusual Equivalence Relation)** Define the relation $\equiv$ on $\mathbb{Z}$ by $m \equiv n$ if and only if 3 divides $m+2n$.

a) Show that $\equiv$ is an equivalence relation
b) Find the equivalence classes?.
Ans: a)
reflexivity: $m \equiv m$ since $3 \mid (m+2m) \Leftrightarrow 3 \mid 3m$
symmetry: If $m \equiv n$ then $3 \mid (m+3n) \Rightarrow \exists k \in \mathbb{Z},\ m+3n = 3k$. But we can write

$$\begin{aligned} n+2m &= (3m+3n)-(m+2n) \\ &= (3m+3n)-3k \\ &= 3(m+n-k) \\ &= 3k_1 \end{aligned}$$

where $k_1 \in \mathbb{Z}$. Hence $3 \mid (n+2m)$ and so $n \equiv m$.
Transitivity: Assume $m \equiv n$ and $n \equiv p$. Hence $\exists k_1, k_2 \in \mathbb{Z}$ such that

$$\begin{aligned} 3 \mid (m+2n) &\Rightarrow m+2n = 3k_1 \\ 3 \mid (n+2p) &\Rightarrow n+2p = 3k_2 \end{aligned}$$

Adding these equations, we have

$$m+3n+2p = 3(k_1+k_2)$$

or

$$\begin{aligned} m+2p &= 3(k_1+k_2)-3n \\ &= 3(k_1+k_2-n) \\ &= 3k_3 \end{aligned}$$

where $k_3 \in \mathbb{Z}$. Hence $3 \mid m+3p$ and so $m \equiv p$ and thus $\equiv$ is transitive.

b) There are three equivalence classes. They are

$$\begin{aligned} [0] &= \{\ldots -9,-6,-3,0,3,6,9,\ldots\} \\ [1] &= \{\ldots -8,-5,-2,1,4,7,10,\ldots\} \\ [2] &= \{\ldots -7,-5,-1,2,5,8,11,\ldots\} \end{aligned}$$

9. **(Equivalence Relation in Calculus)** In the set of continuous functions $C[0,1]$ defined on the closed interval $[0,1]$ define the relation $R \in C[0,1] \times C[0,1]$ by $f\,R\,g$ if and only if

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx \ .$$

a) Show that R is an equivalence relation.
Ans: Simple property of integrals.

b) Find a function $g \in C[0,1]$ equivalent to $f(x) = x$ but $f \neq g$.
Ans: $g(x) = 1 - x,\ 0 \le x \le 1$.

10**. (More Equivalence Relations in Analysis)** Let $X = [-1,1]$ and define an equivalence relation R on X by xRy iff $x^2 = y^2$, $x, y \in X = [-1,1]$. Describe the quotient set X / R .

Ans: Every $x \in [-1,1]$ is equivalent to itself and $-x$. In other words. 1 is equivalent to 1 and -1. -1/3 is equivalent to -1/3 and 1/3, 0 is equivalent to 0, and so on. Hence, the quotient set is

$$X / R = [-1,1] / R = \{\{x, -x\} : -1 \le x \le 1\}$$

As a set $R \subseteq X \times X = [-1,1] \times [-1,1]$ the relation is $R = \{(x,x) \cup (x,-x) : -1 \le x \le 1\}$ which consists of the two lines $y = x$ and $y = -x$ for $x \in [-1,1]$.

11. **(Equivalence Sets of Polynomials)** Define $\mathbb{C}[x]$ as the set of polynomials in real variable x and $I \subseteq \mathbb{C}[x]$ as the subset of polynomials that satisfy $p(0) = 0$. Show that for $f, g \in \mathbb{C}[x]$ the relation $f \equiv g$ defined by

$$f \equiv g \Leftrightarrow f - g \in I$$

is an equivalence relation[1].

Ans: Reflexive: Clearly $f \equiv f$ since $f - f = 0$ which satisfies $(f - f)(0) = 0$
Symmetric:

$$f \equiv g \Rightarrow (f - g)(0) = 0 \Rightarrow f(0) = g(0) \Rightarrow g(0) = f(0) \Rightarrow (g - f)(0) \Rightarrow g \equiv f$$

[1] In the language of abstract algebra, the set $\mathbb{C}[x]$ is a polynomial ring and the subset I a vanishing ideal in the ring.

Transitive: If $f \equiv g, g \equiv h$ we have $f(0) = g(0)$ and $g(0) = h(0)$, hence $f(0) = h(0)$ which implies $f \equiv h$. Hence the relation is transitive.

12. **(Modular Arithmetic)** Let $x, y, n \in \mathbb{Z}$ we say x is congruent to y mod n, written $x \equiv y (\operatorname{mod} n)$, if and only if $x - y$ is divisible by n. Prove that the relation $\equiv$ is an equivalence relation.

Ans:

Reflexive: Clearly $x \equiv y(\operatorname{mod} n)$ since any integer n divides $x - x = 0$

Symmetric: Assume $x \equiv y(\operatorname{mod} n)$. Hence n divides $x - y$ which means $\exists q \in \mathbb{Z}$, $x - y = qn$. But then $y - x = (-q)n$ and so n divides $y - x$ and thus $y \equiv x(\operatorname{mod} n)$. Thus $\equiv$ is symmetric.

Transitive:Assume $x \equiv y(\operatorname{mod} n)$ and $y \equiv z(\operatorname{mod} n)$.Therefore $\exists q_1, q_2 \in \mathbb{Z}$ so that $x - y = q_1 n,\ y - z = q_2 n$. Adding these equations gives

$$(x - y) + (y - z) = x - z = (q_1 + q_2)n = q_3 n$$

where $q_3 = q_1 + q_2 \in \mathbb{Z}$. Hence n divides $x - z$ which means $x \equiv z(\operatorname{mod} n)$. Hence the relation $\equiv$ is transitive. Thus we have $\equiv$ is an equivalence relation.

13. **(An Old Favorite)** The equals relation "$=$" is the most familiar equivalence relation. What are the equivalence classes of the equals relation on the set $A = \{1, 2, 3, 4, 5\}$?

Ans: The equivalence classes are $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$. In other words each member of the set is its own equivalence class.

14. **(Equivalence Classes in Logic)** Define an equivalence relation on logical sentences by saying two sentences are equivalent if they have the same truth value. Find the equivalence classes in the following collection of sentences.

a) $1 + 2 = 3$
b) $3 < 5$
c) $2 | 7$
d) $x^2 < 0$ for some real number.
e) $\sin^2 x + \cos^2 x = 1$
f) Georg Cantor was born in 1845.
g) Leopold Kronecker was a big fan of Cantor.
h) Cantor's theorem guarantees larger and larger infinite sets.

Ans: $\text{true} = \{a, b, e, f, h\}, \text{false} = \{c, d, g\}$

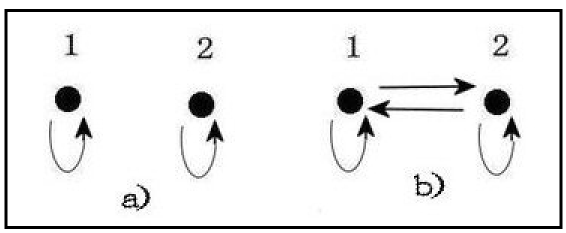
15. **(Similar Matrices)** Two square matrices A, B are equivalent if there is an invertible matrix M such that $MAM^{-1} = B$. Show that similarity of matrices is an equivalence relation.

Ans: Similar matrices can be thought of as describing the same linear mapping, but with respect to different bases.

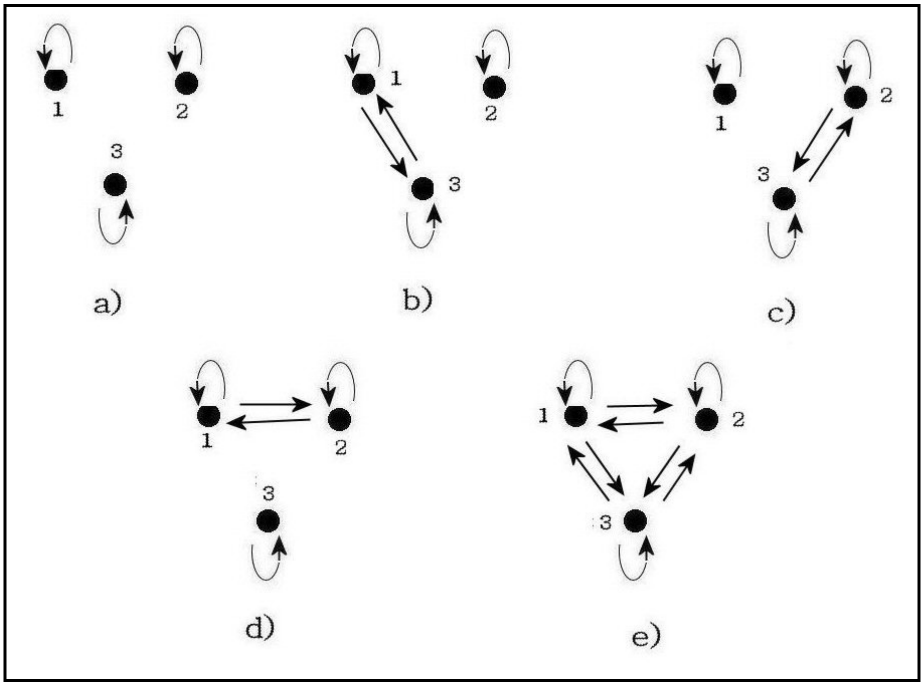
Reflexive A is equivalent to itself since $IAI^{-1} = A$
Symmetric Assume A is equivalent to B. Hence, $MAM^{-1} = B$ for some invertible matrix M. Hence we can write $A = M^{-1}BM$ which says B is equivalent to A.

Transitive Assume A is equivalent to B and that B is equivalent to C. Hence, there exists matrices M_1, M_2 so that $M_1AM_1^{-1} = B$ and $M_2BM_2^{-1} = C$. Hence $M_1AM_1^{-1} = B = M_2^{-1}CM_2$. Hence, carrying out some basic manipulations on matrices, we find $(M_2M_1)A(M_2M_1)^{-1} = C$ which says A is equivalent to C.

16. **(Counting Equivalence Relations)**
a) Count the number of different equivalence relations on the set $A = \{1,2\}$.
b) Count the number of different equivalence relations on the set $A = \{1,2,3\}$
Ans: a) The two equivalence relations on $A = \{1,2\}$ are



b) The five equivalence relations on $A = \{1,2,3\}$ are



17. (**Arithmetic in Modular Arithmetic**) Suppose

$$a \equiv c \pmod{5}$$
$$b \equiv d \pmod{5}$$

Show

a) $a+b \equiv c+d \pmod{5}$
b) $a-b \equiv c-d \pmod{5}$
c) $ab \equiv cd \pmod{5}$

Ans: Student Project

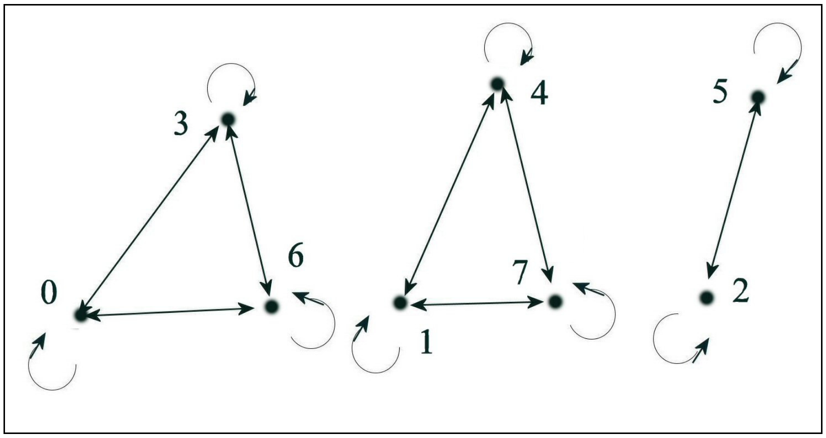
18. **(Maping into the Equivalence Class)** Let X denote the students at your college or university and define the equivalence relation as "being in the same class (freshman, sophomore, junior or senior). Define the mapping $f : x \to [x]$ that sends each student $x \in X$ into his or her residue class $[x]$. Is this a well-defined function? What is your value under this mapping?

Ans: The function is a well-defined function from X to the quotient set of $X \bmod R$, where R is the equivalence relation of "being in the same class," and the value of the mapping when $x = \text{yourself}$ is your class ranking as freshman, sophomore, junior, or senior.

19. **(Equivalence Classes as Digraphs)** Inasmuch as equivalence classes are binary relations, they can be represented by digraphs. Draw digraphs that represent the

equivalence classes of the set $A=\{0,1,2,3,4,5,6,7\}$ when two elements are equivalent if they have the same remainder when divided by 3.

Ans:



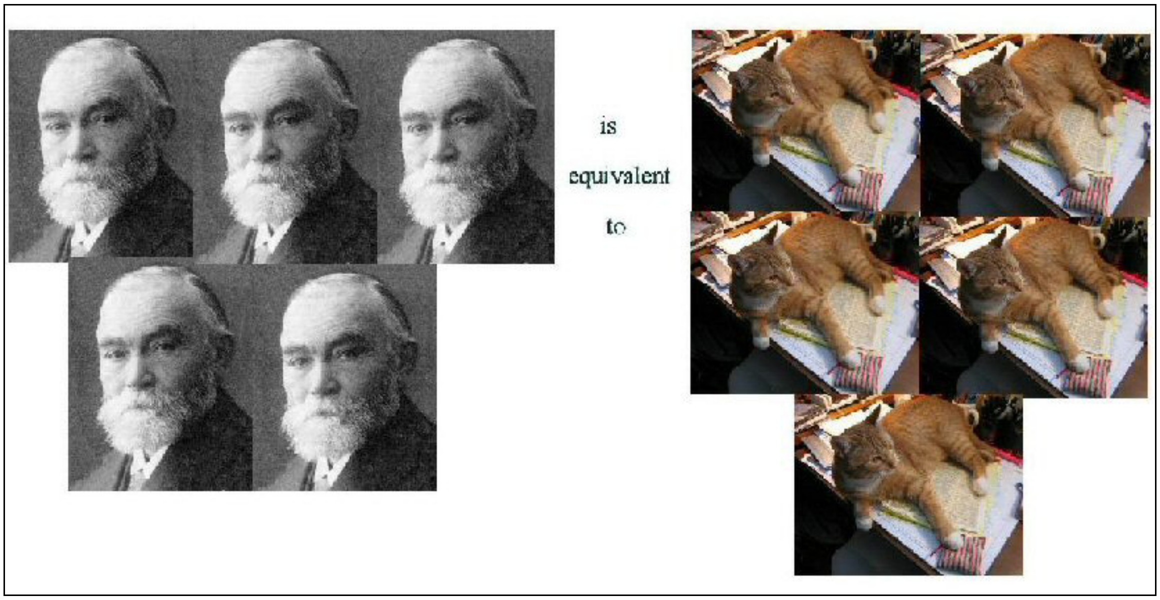
20.(**Inventing New Numbers from Equivalence Classes**) It is possible to *define* the negative integers and zero from the natural numbers by means of equivalence classes of pairs of natural numbers. The idea is to define the negative numbers and zero as the collection of equivalent pairs of natural numbers (a,b) with $a<b$, where two pairs (a,b) and (c,d) are equivalent i.e. $(a,b)\equiv(c,d)$, if and only if $a+d=b+c$. List the different equivalence classes for $\mathbb{N}\times\mathbb{N}$ and illustrate which equivalence classes can be associated with the negative integers and zero.

Ans:

$$
\begin{aligned}
[0]&=\{(n,n):n\in\mathbb{N}\}=\{(1,1),(2,2),(3,3),,,\}\\
[-1]&=\{(n,n+1):n\in\mathbb{N}\}=\{(1,2),(2,3),(3,4),,,\}\\
[-2]&=\{(n,n+2):n\in\mathbb{N}\}=\{(1,3),(2,4),(3,5),,,\}\\
[-3]&=\{(n,n+3):n\in\mathbb{N}\}=\{(1,4),(2,5),(3,6),,,\}\\
&\quad\ldots\qquad\ldots\qquad\ldots\\
[-k]&=\{(n,n+k):n\in\mathbb{N}\}=\{(1,1+k),(2,2+k),(3,3+k),,,\}\\
&\quad\ldots\qquad\ldots\qquad\ldots
\end{aligned}
$$

We now have to define the usual properties of arithmetic for these new numbers and show they satisfy the usual properties.

21. (**Find the Equivalence Relation**) We say the following are equivalent. What is the equivalence relation?



Ans: Having the same cardinality. Two sets are equivalent if their members can be put in a 1-1 correspondence with each other.

22. **(Counting Partitions)** Find the different partitions of the sets

a) $A=\{1,2\}$
b) $A=\{1,2,3\}$

Ans:

a) There are two partitions of $A=\{1,2\}$ and they are

$$P_1=\{\{1,2\}\},\ P_2=\{\{1\},\{2\}\}$$

b) The five partitions of $A=\{1,2,3\}$ are

$$P_1=\{\{1\},\{2\},\{3\}\}$$
$$P_2=\{\{1\},\{2,3\}\}$$
$$P_3=\{\{2\},\{1,3\}\}$$
$$P_4=\{\{3\},\{1,2\}\}$$
$$P_5=\{\{1,2,3\}\}$$

The number of partitions of a set of size n is called the nth Bell number. More information about them can be found on the internet.

23. **(Interesting Equivalence Relation)** Let $A=\{0,1,2,3,\ldots,29,30\}$ be the set of nonnegative integers from 0 to 30, and define the relation R on A by $m\equiv n$ if and only

if the product of the digits of m is the same as the product of the digits of n. For example are equal $16 \equiv 23, \; 4 \equiv 14$.

a) Show that $\equiv$ is an equivalence relation on A.

b) Find the equivalence classes of the relation.

Ans: a) $m \equiv m$ is obvious, $m \equiv n \Rightarrow n \equiv m$ is obvious, $m \equiv n \wedge n \equiv p \Rightarrow m \equiv p$ is obvious.

b) The equivalence classes are listed below.

Product	**Integers**
0	0,10,20,30
1	1,11
2	2,12,21
3	3,13
4	4,14,22
5	5,15
6	6,16,23
7	7,17
8	8,18,24
9	9,19
10	25
12	26
14	27
16	28
18	29

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