Section 3.4: The Function Relation

1 **(Testing Relations)** Which of the following relations are functions? For functions what is the domain and range of the function?

a) $R=\{(1,3),(3,4),(4,1),(2,1)\}$ **Ans:** yes

b) $R=\{(1,3),(1,4),(1,2),(3,1)\}$ **Ans:** no

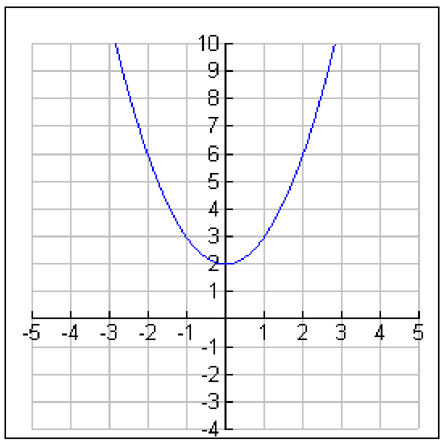
c) $R=\{(1,3),(3,4),(1,1)\}$ **Ans:** no

d) $R=\{(1,2),(2,2),(3,2),(2,3)\}$ **Ans:** no

2. **(Graphing Relations and Functions)** Graph each of the following relations R on $\mathbb{R}$ and tell which relations are functions.

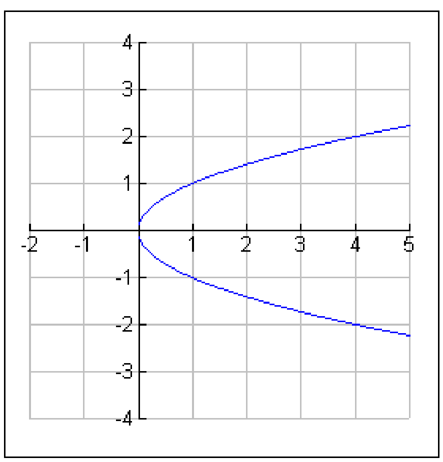
a) $R=\{(x,y): y=x^2\}$

Ans: function



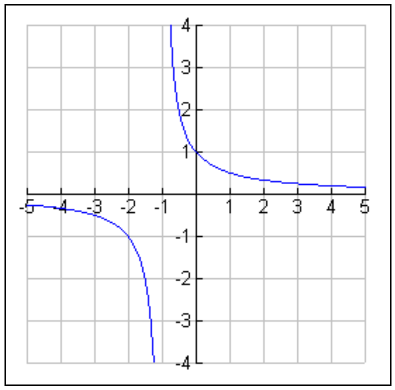
b) $R=\{(x,y): y=\pm\sqrt{x}\}$

Ans: not a function



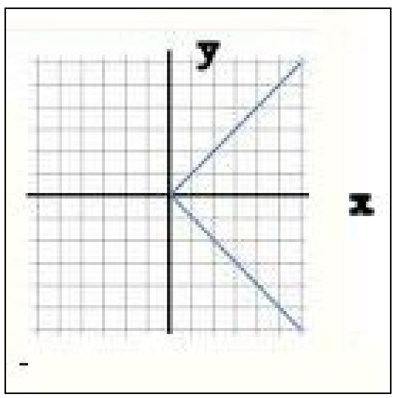
c) $R=\left\{(x, y): y=\frac{1}{x+1}\right\}$

Ans: **f**unction



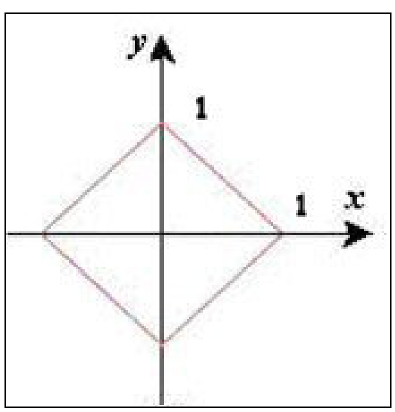
d) $R=\{(x, y): x=|y|\}$

Ans: not a function



e) $R=\{(x, y):|x|+|y|=1\}$

Ans: not a function



3. **(Find the Mystery Function)** Find a function that "tears" the interval $[0,1]$ into two parts at its midpoint and then "stretches" each part uniformly to twice its length.

Ans: $$f(x)=\begin{cases} 1+2x, & \frac{1}{2}\le x\le 1 \\ -2+2x, & 0<x<\frac{1}{2} \end{cases}$$

4. Find $f\circ g$ and $g\circ f$ and their domains for the given functions f,g where the domains of the functions are assumed to be all values for which the function is well-defined.

a) $$f(x)=\{(-2,3),(-1,1),(0,0),(1,-1),(2,-3)\}$$
$$g(x)=\{(3,1),(0,2),(-1,-2),(2,0),(-3,1)\}$$

Ans:
$$f\circ g=\{(3,-1),(0,-3),(-1,3),(2,0),(-3,-1)\}$$
$$g\circ f=\{(-2,1),(0,2),(1,-2),(2,1)\}$$

b) $f(x)=2x+3,\ g(x)=-x^2+5$

Ans:
$$(f\circ g)(x)=f[g(x)]=f(-x^2+5)=2(-x^2+5)+3=-2x^2+13$$
$$(g\circ f)(x)=g[f(x)]=g(2x+3)=-(2x+3)^2+5=-4x^2-12x-4$$

c) $f(x)=\dfrac{1}{x^2+1},\ g(x)=x^2$

Ans:
$$(f\circ g)(x)=f[g(x)]=f(x^2)=\frac{1}{(x^2)^2+1}=\frac{1}{x^4+1}$$
$$(g\circ f)(x)=g[f(x)]=g\left(\frac{1}{x^2+1}\right)=\left(\frac{1}{x^2+1}\right)^2$$

d) $f(x)=|x|,\ g(x)=|x|$

Ans:
$$(f\circ g)(x)=f[g(x)]=f(|x|)=\big||x|\big|=|x|$$
$$(g\circ f)(x)=g[f(x)]=g(|x|)=\big||x|\big|=|x|$$

e) $f(x)=\sqrt{x},\ g(x)=x-2$

Ans:

$$(f\circ g)(x)=f[g(x)]=f(x-2)=\sqrt{x-2}.\ x\geq 2$$
$$(g\circ f)(x)=g[f(x)]=g(\sqrt{x})=\sqrt{x}-2,\ x\geq 0$$

f) $f(x)=\sqrt{1-x^2},\ g(x)=\sqrt{x^2-1}$

Ans: Student Project

g) $f(x)=\sqrt{x+2},\ g(x)=x^2-2$

Ans:

$$(f\circ g)(x)=f[g(x)]=f(x^2-2)=\sqrt{(x^2-2)+2}=\sqrt{x^2}=|x|,\ x\geq 0$$
$$(g\circ f)(x)=g[f(x)]=g(\sqrt{x+2})=(\sqrt{x+2})^2-2=(x+2)-2=x,\ \ x\geq -2$$

5. **(Composition of Three Functions)** Given $S=\{1,2,3,4\}$ and functions f,g,h all defined from S to S, find the composition $f\circ(g\circ h)$ when

$$f=\{(1,3),(2,4),(3,1),(4,2)\}$$
$$g=\{(1,2),(2,2),(3,4),(4,3)\}$$
$$h=\{(1,4),(2,4),(3,1),(4,3)\}$$

Ans:

$$(g\circ h)(x)=\{(1,3),(2,3),(3,2),(4,4)\}$$
$$(f\circ(g\circ h))(x)=\{(1,3),(2,1),(3,4),(4,2)\}$$

5. **(Backwards Compositions)** Often one can interpret a given function h as the composition of two functions. For the following functions h, determine two functions f,g whose composition $f\circ g$ is $h(x)$.

a) $h(x)=(x-1)^2+(x-1)+3$

Ans: $f(x)=x^2+x+3,\ g(x)=x-1$. One could also write

$$h(x)=(x-1)^2+(x-1)+3=x^2-x+5=(x+1)^2-3(x+1)+7$$

and write the composition of the functions $f(x) = x^2 - 3x + 7,\ g(x) = x + 1$ so one should realize there are an infinite number of different ways to represent this composition.

b) $h(x) = \sin(1/x)$

Ans: $f(x) = \sin x,\ g(x) = 1/x$

c) $h(x) = x^2 + x + 1$

Ans: We can write $h(x) = x^2 + x + 1 = (x+1)^2 - (x+1) + 1$ which can be represented by the composition $f \circ g$ where $f(x) = x^2 - x + 1,\ g(x) = x + 1$. There are obviously an infinite number of ways to write h as a composition of two functions.

d) $h(x) = e^{x^2} + 1$

Ans: $f(x) = e^x + 1,\ g(x) = x^2$

6. **(Decomposing a Function as a Composition)** Write the function $h(x) = x^2 + 1$ as a composition $h = f \circ g$ of two functions in an infinite number of different ways.

Ans: If we write the function as

$$h(x) = x^2 + x + 1 = (x-a)^2 + 2a(x-a)x + (a^2 + 1)$$

we see it is the composition $f \circ g$ where

$$f(x) = x^2 + 2ax + (a^2 + 1),\ g(x) = x - a .$$

But a can be any real number we have represented h as the composition of two functions in an infinite number of ways.

7. **(Classroom Function)** Let A be the set of students in your Intro to Abstract Math Class and B be the natural numbers from 1 to 100.

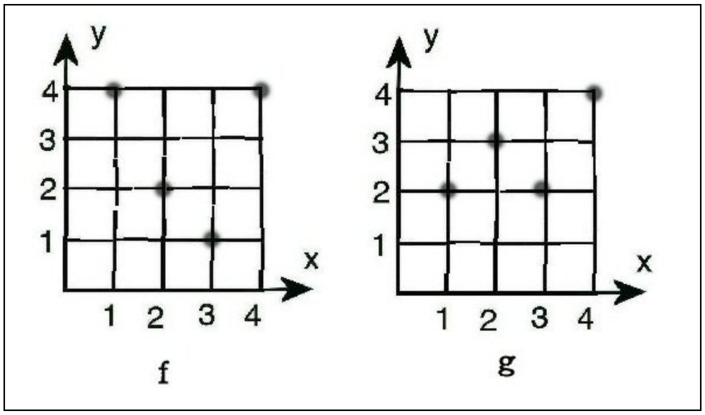
a) Suppose we assign to each student in the class the age of that student. That is, if x is a student in the class, then $f(x)$ is the age of x. Is this a function from A to B?

b) Suppose we assign to each natural number $n \in B$ all students in A whose age is n. Is this a function from B to A?

a) **Ans:** yes, unless there is someone in the class over 100

b) Ans: If there are two or more students in the class the same age it is not a function. Also for the domain to be B, there would have to be a student in the class of every age from 1 to 100, which is very doubtful. Hence, we can safely say the described relation on $B \times A$ is not a function.

8.(**More Compositions**) Given functions f, g each with domain and codomain $A = \{1,2,3,4\}$ as illustrated in the following figure, find the following.



a) $f \circ g$ **Ans:** $f \circ g = \{(1,2),(2,1),(3,2),(4,4)\}$

b) $g \circ f$ **Ans**: $g \circ f = \{(1,4),(2,3),(3,2),(4,4)\}$

c) $f \circ f$ **Ans:** $f \circ f = \{(1,4),(2,2),(3,4),(4,4)\}$

d) $g \circ g$ **Ans:** $g \circ g = \{(1,3),(2,2),(3,3),(4,4)\}$

9. **(Shifting Domain of a Composition)** Given the function

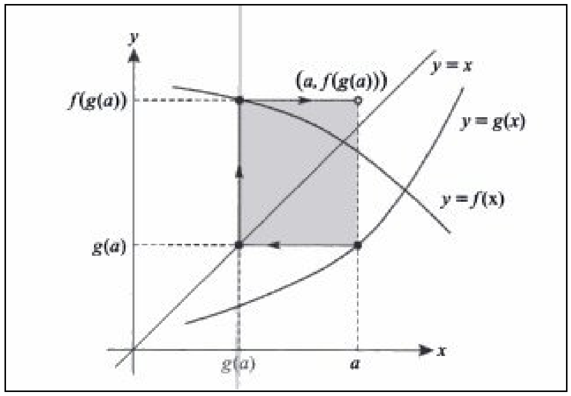
$$f(x) = \frac{1}{1-x}, \quad x \neq 1$$

whose domain is all real numbers except 1. Now find the domain of $f \circ f$.
Ans:

$$(f \circ f)(x) = \frac{1}{1-\left(\frac{1}{1-x}\right)} = \frac{1-x}{(1-x)-1} = \frac{x-1}{x}, \quad x \neq 0$$

Hence, the domain is all real numbers except 0 or 1.

10. (**Graphing a Composition**) Draw an arbitrary graph of two real-valued functions of a real variable f, g and select an arbitrary real number x. Illustrate on the graph the location of $(f \circ g)(a)$.
Ans:



11. **(Compositions)** Find the composition $f \circ g$ of the functions $f : \mathbb{R} \to \mathbb{R}^3$, $g : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = (t, t^2, t^3)$, $g(t) = \sin t$.

Ans: $(f \circ g)(x) = f\left[g(x)\right] = f(\sin t) = \left(\sin t, \sin^2 t, \sin^3 t\right)$

12. **(Composition of Operators)** Define operators $L_1(f) = xf(x) + 1$, $L_2(f) = x^2 \dfrac{df}{dx}$ on domains sufficient for all operators to exist. Find

a) $L_1 \circ L_2$
b) $L_2 \circ L_1$

a) **Ans**: $L_1 \circ L_2 = L_1\left[L_2(f)\right] = L_1\left(x^2 \dfrac{df}{dx}\right) = x\left(x^2 \dfrac{df}{dx}\right) + 1 = x^3 \dfrac{df}{dx} + 1$

b) **Ans:**

$$L_2 \circ L_1 = L_2\left[L_1(f)\right] = L_2\left(xf(x) + 1\right) = x^2\left[\frac{d\left(xf(x)+1\right)}{dx}\right] = x^2\left[xf'(x) + f(x)\right]$$

13. **(Functions from Everyday Life)** Ann has gotten a summer job selling subscriptions to an internet service. She receives a weekly salary of \$500 plus a 6% commission on sales over \$5000. Assuming she sells enough to get her commission, write Ann's weekly salary as a composition of two functions.

Ans: Ann's total salary will be

$$s(x) = \begin{cases} 500 & x \le 5000 \\ 0.06(x - 5000) + 500 & x > 5000 \end{cases}$$

so if she sells more than \$5,000 in subscriptions she will earn $s(x)=0.06(x-5000)+500$ which we can write as the composition $s(x)=(f\circ g)(x)=f\left[g(x)\right]$ where

$$f(x)=0.06x+500,\ g(x)=x-5000.$$

14. **(Recursive Functions)** A recursive function is one that is defined in terms of itself, normally defined over a restricted subset of its domain. For example, the factorial function $n!=n(n-1)(n-2)\cdots(2)(1)$ can be defined **recursively** as

$$n!=\begin{cases}0 & n=1\\ n(n-1)! & n>1\end{cases}$$

The greatest common divisor of two positive integers m and n, which is the largest positive integer that divides both m and n, can be interpreted as a function $\gcd:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ defined recursively by

$$\gcd(m,n)=\begin{cases}m & \text{if } n=0\\ \gcd(n,\text{remainder}(m,n)) & \text{if } m\geq n \text{ and } n>0\end{cases}$$

Use this recursive definition to find the greatest common divisor of the following numbers.

a) $m=25,\ n=5$

b) $m=56,\ n=2$

c) $m=37,\ n=3$

Ans:

a) $\gcd(25,5)=\gcd(5,0)=5$

b) $\gcd(56,2)=\gcd(2,0)=2$

c) $\gcd(37,3)=\gcd(3,1)=\gcd(1,0)=1$

15. **(Functional Equation)** A **functional equation** is an equation which expresses implicitly the value of the function at a point in terms of the value of the function at a different point or points. Below are listed four well-known functional equations. Find the function or functions in explicit form that satisfies the given functional equations.

a) $f(x+y)=f(x)+f(y)$ Cauchy's functional equation

a) $f(x+y)=f(x)f(y)$

b) $f(xy) = f(x) + f(y),\ x, y > 0$

c) $f(xy) = f(x)f(y),\ x, y > 0$

Ans:

a) satisfied by linear functions; that is $f(x) = cx,\ c$ arbitrary constant

b) satisfied by exponential functions, that is $f(x) = e^{cx},\ c$ arbitrary constant

c) satisfied by logarithmic functions; that is $f(x) = c\log(x),\ c$ arbitrary constant

d) satisfied by power functions: that is $f(x) = x^c,\ c$ arbitrary constant, or $f(x) = 0$

16. **(Injections, Surjections, Bijections)** Find functions f_1, f_2, f_3, f_4 from $\mathbb{N}$ to $\mathbb{N}$ that satisfy
the following properties.

a) f_1 is neither 1-1 or onto.

Ans: a) $f_1(n) = 1,\ \forall n \in \mathbb{N}$

b) f_2 is 1-1 but not onto.

Ans: b) $f_2(n) = \begin{cases} 1, & n = 1 \\ n+1, & n = 2, 3, \ldots \end{cases}$

c) f_1 is onto but not 1-1.

Ans: c) $f_3(n) = \begin{cases} (n+1)/2 & n \text{ odd} \\ n/2 & n \text{ even} \end{cases}$

d) f_1 is both 1-1 and onto.

Ans: d) $f_4(n) = n$

17. Find examples of the following functions f.

a) f maps $\mathbb{R}$ to $\{1, 2, 3\}$

Ans: a) $f(0) = 1,\ f(1) = 2,\ f(x) = 3,\ x \in \mathbb{R} - \{0, 1\}$

b) f maps $\mathbb{N}$ to $\mathbb{R}$

Ans: b) $f(n) = n,\ n \in \mathbb{N}$

c) f maps $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$

Ans: $f(x, y) = xy,\ x \in \mathbb{R}, y \in \mathbb{R}$

d) f maps $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$

Ans: $f(t)=(t,t^2),\ t\in\mathbb{R}$

e) f maps $\{a,b,c\}$ to $[0,1]$

Ans: $f(a)=1,\ f(b)=0,\ f(c)=1$

18. **(Injections, Surjections, and Bijections)** Which of the following functions $f:\mathbb{R}\to\mathbb{R}$ are injective, surjective, and bijective. Take the domains of the functions as those subsets of $\mathbb{R}$ for which the function is well-defined.

a) $f(x)=x^3-2x+1$ **Ans:** surjective

b) $f(x)=\sin(1/x)$ **Ans:** none of the three

c) $f(x)=\begin{cases} x^2 & x\le 0 \\ x+1 & x>0 \end{cases}$ **Ans:** none of the three

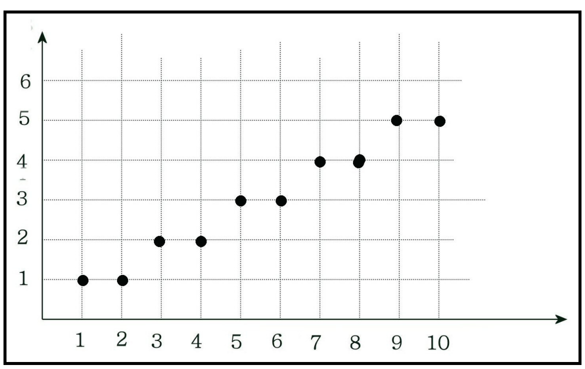
d) $f(x)=e^{-x}$ **Ans:** injective

19. **(Interesting Function)** Let $f:\mathbb{N}\to\mathbb{N}$ be the function defined by: the even numbers $2n\to n$, the odd numbers $2n-1\to n$.

a) Draw part of the graph of this function.
b) Is this function 1-1 ?
c) Is this function an onto function ?

Ans:

a)



b) not 1-1

c) onto $\mathbb{N}$

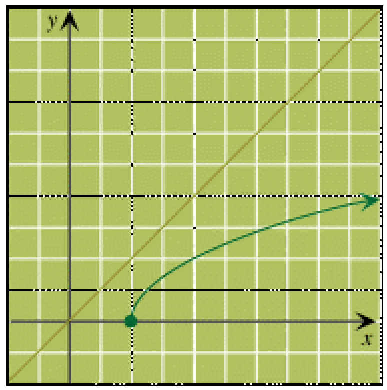
20. **(Inverse Function)** Given the function

$$f(x)=\sqrt{x-2},\ x\ge 2$$

a) Draw the graph of f

b) Find the domain and range of f .
c) Prove that the function is 1-1.
d) Find the inverse of the function.
e) Find the domain and range of the inverse function.
f) Draw the graph of the inverse function.

Ans: a)



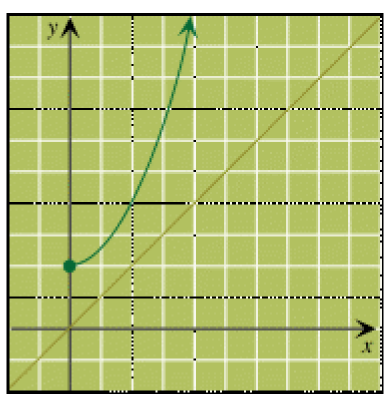
Ans: b) $\text{dom}(f)=[2,\infty)$, $\text{rng}(f)=[0,\infty)$

Ans: c) Setting $\sqrt{x-2}=\sqrt{y-2}$ for $x, y \geq 2$ and squaring we find $x-2=y-2 \Rightarrow x=y$

Ans: d) Setting $y=\sqrt{x-2} \Rightarrow y^2=x-2 \Rightarrow x=y^2+2$. Hence $f^{-1}(x)=x^2+2$.

Ans: e) $\text{dom}(f^{-1})=[0,\infty)$, $\text{rng}(f^{-1})=[2,\infty)$

Ans: f) The graph of the inverse function is the mirror image of the graph of the function reflected in the straight line $y=x$.



21. **(Function as Ordered Pairs)** Given the function $f:\{1,2,3\} \to \mathbb{N}$ defined by $f=\{(1,3),(2,5),(3,1)\}$:

a) Is f 1-1?
Ans: yes

b) Is f onto?
Ans: no

c) What is the range of f ?
Ans: $\operatorname{rng}(f)=\{3,5,1\}$

22**. (1-1 but not Onto)** Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ that is 1-1 but not onto.

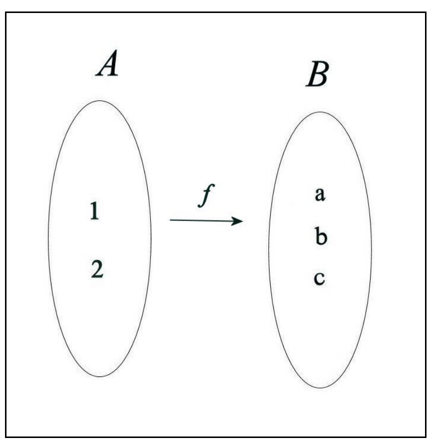
Ans: $f(x)=\begin{cases} x & x\geq 0 \\ x-1 & x<0 \end{cases}$

23. **(Hmmmmmmmmm)** For what value of the exponent $n\in\mathbb{N}$ is the function $f(x)=x^n$ an injection?
Ans: odd values

24. **(Prove or Find a Counterexample)** Is it true that if a function is 1-1, then its inverse is also 1-1? If so prove it, if not find a counterexample.

25**. Counting Functions I)** Let $A=\{1,2\}$, $B=\{a,b,c\}$ as shown in the diagram.

a) How many functions are there from A to B ?
b) How many injections are there from A to B ?
c) How many surjections are there from A to B ?
d) How many bijections are there from A to B ?



Ans (functions): The number 1 can map into one of 3 letters and 2 can map into 3 letters, … and so using the multiplication principle, there are $3^2 = 9$ functions.

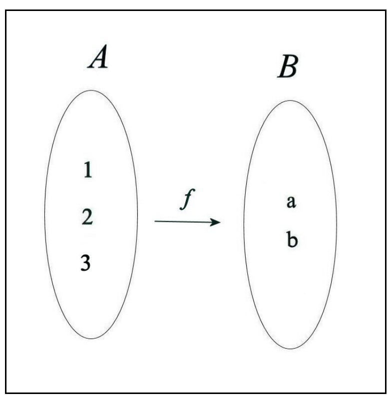
Ans (injections): There are $3 \times 2 = 6$ injections. If you leave out c from in image then there are 2! permutations for mapping $\{1,2\}$ to $\{a,b\}$. But you can also leave out b or c so the total number of injections is $3 \times 2! = 6$

Ans (surjections): There are no onto maps in the case the cardinality of the codomain is greater than the cardinality of the domain.

Ans (bijections): There are no bijections. The cardinality of A and B would have to be the same in order that there be a bijection.

26. **(Counting Functions II)** Let $A = \{1,2,3\}$, $B = \{a,b\}$ as shown in the diagram.

a) How many functions are there from A to B?
b) How many injections are there from A to B?
c) How many surjections are there from A to B?
d) How many bijections are there from A to B?



Ans (functions): The number 1 can map into one of 2 letters, 2 can map into 2 letters, … and so using the multiplication principle, there are $2^3 = 8$ functions.

Ans (injections): There are no injections. In order that there be injections from A to B the cardinality of A must be less than or equal to the cardinality of B.

Ans (surjections): In order that the mapping be onto one of the letters $a,b \in B$ has to be mapped onto two times. Suppose $a \in B$ is mapped onto by two of the three numbers 1,2,3. There are $\binom{3}{2} = 3$ ways for this to happen. So the total number of injections is $2\binom{3}{2} = 6$ surjections. Can you write them down?

Ans (bijections): There are no bijections. The cardinality of A and B would have to be the same in order that there be a bijection.

27. **(Finding Injections and Surjections)**

a) Find a function from the natural numbers to themselves that is injective but not surjective.
Ans: $f(n) = n^2$

b) Find a function from the natural numbers to themselves that is surjective but not injective.
Ans:

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & n = 1,3,5,\dots \\ \dfrac{n}{2}, & n = 2,3,4,\dots \end{cases}$$

In other words $f(1) = f(2) = 1,\ f(3) = f(4) = 2, f(5) = f(6) = 3, \dots$.

28. **(Composition of Onto Functions)** Prove that if g maps X onto Y, and f maps Y onto Z, then the composition $f \circ g$ maps X onto Z. In short, the composition of two surjections is a surjection.

Ans: Choose any $z \in Z$. Since f is onto Z there exists a $y \in Y$ such that $f(y) = z$. But g is onto Y so there exists an $x \in X$ such that $g(x) = y$. Thus $(f \circ g)(x) = f[g(x)] = f(y) = z$ and so $f \circ g : X \to Z$ is onto.

29. **(Composition of 1-1 Functions)** Prove that if g is a 1-1 mapping from X to Y, and f is a 1-1 mapping from Y to Z, then the composition $f \circ g$ is a 1-1 mapping from X to Z. In other words, the composition of injections is an injection[1].

Ans: Suppose $x, x' \in X$ and that $x \neq x'$. Since g is 1-1 we know $g(x) \neq g(x')$. But f is also 1-1 and so we have

$$(f \circ g)(x) = f[g(x)] \neq f[g(x')] = (f \circ g)(x')$$

which shows that $f \circ g$ is 1-1.

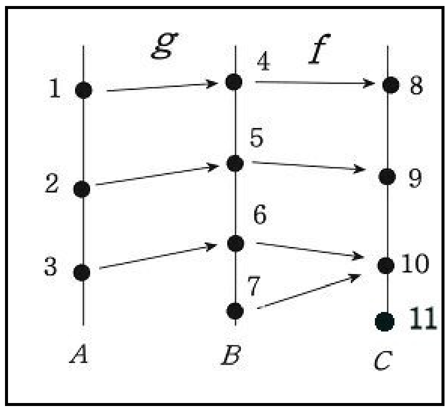
[1] Problem 15 and 14 show that the composition of two bijections is a bijection.

30. **(Hmmmmmmmmmm)** Let $g : A \to B, f : B \to C$ and $f \circ g : A \to C$. Find examples of the following:

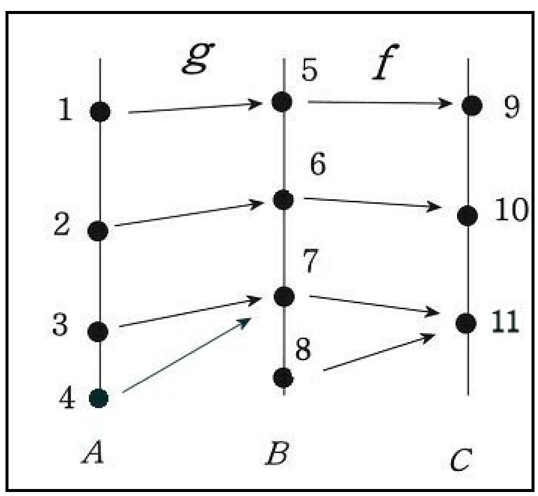
a) A 1-1 composition $f \circ g : A \to C$ where $f : B \to C$ is not 1-1.
b) An onto composition $f \circ g : A \to C$ where $g : A \to B$ is not onto.
c) A bijection composition $f \circ g : A \to C$ where g is not onto and f is not 1-1.

Ans:

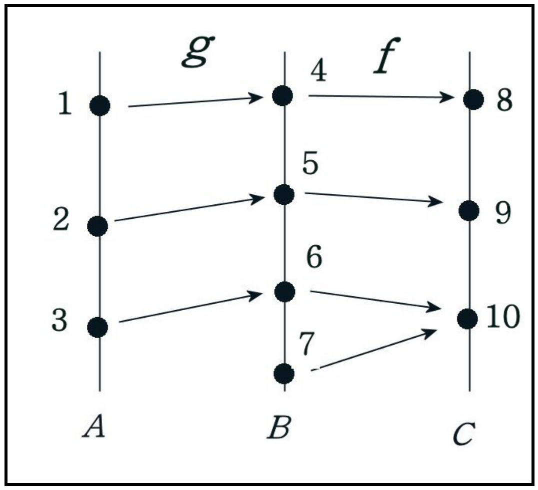
a) A 1-1 composition $f \circ g : A \to C$ where $f : B \to C$ is not 1-1.



b) An onto composition $f \circ g : A \to C$ where $g : A \to B$ is not onto.



c) A bijection composition $f \circ g : A \to C$ where g is not onto and f is not 1-1.



31. **(More Counting Functions)** Let $S = \{1, 2, 3, \ldots, n\}$.

a) How many functions are there from S to S?
b) How many onto functions are there from S to S?
c) How many 1-1 functions are there from S to S?
d) How many bijective functions are there from S to S?

Ans: a) n^n Every value 1,2,…, n in the domain can have one of n values.

Ans: b), c), d) In this case, injections, surjections and bijections are all the same, and their number is the number of permutations of $\{1, 2, 3, \ldots, n\}$, namely $n!$

32**. (Counting Functions in General)** If A is a set with m elements and B is a set with n elements how many functions $f : A \to B$ of the different types are there?

a) All functions
b) All 1-1 functions
c) All bijections

a) **Ans** (functions): Since for each of the n elements of A there are m possibilities for values of f. Hence there are n^m possible functions.

b) **Ans** (1-1) If $m < n$ (more points in B than in A) then to count 1-1 functions we take into account that for any $b \in B$ we pick as an image of some $a \in A$ may not be chosen that value again. Hence, the number of 1-1 functions is

$$n \times (n-1) \times (n-2) \times \cdots \times (n-m+1) = \frac{n!}{(n-m)!}$$

However if $m > n$ (more points in A than B) then there are no injective functions (this is called the pigeon hole principle). If $m = n$ then an injective function is automatically surjective as well as bijective and the number of these functions is $n!$ (or $m!$).

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c) **Ans** (bijections) The number of bijections is zero if $m \neq n$. However, if $m = n$, then the number of bijections is simply the number of permutations on a set of $m = n$ elements or $m!$

33**. (Cantor-Bernstein Theorem)** Let A, B be two sets. The **Cantor–Bernstein theorem** states that if there exists an injection[2] $f : A \to B$ and an injection $g : B \to A$, then there is a bijection $h : A \to B$. In terms of the cardinality of the sets A, B this means that if $|A| \le |B|$ and $|B| \le |A|$ then $|A| = |B|$. In other words, A and B have the same cardinality. Use the Cantor-Bernstein theorem to show the sets $(0,1)$ and $[0,1]$ have the same cardinality.

Ans: The mapping $f : (0,1) \to [0,1]$ defined by $f(x) = x$ is clearly an injection from $(0,1)$ to $[0,1]$ (not onto but 1-1). Also the mapping $g : [0,1] \to (0,1)$ defined by

$$g(x) = \frac{1}{2} + \frac{x}{10}$$

is an injection from $[0,1]$ to $(0,1)$ (not onto but 1-1). Hence by the Cantor-Bernstein theorem there is a bijection from $[0,1]$ to $(0,1)$, or vice versa and so $[0,1]$ and $(0,1)$ have the same cardinality, which we know to be the cardinality of the continuum c.

34. **(Euler Totient Function)** In number theory the **Euler totient function** $\phi(n)$ (or **phi function**) is a function defined on the natural numbers $\phi : \mathbb{N} \to \mathbb{N}$ which is the number of natural numbers less than n that are **coprime** with n, where a number is coprime with another if the greatest common divisor of the two numbers is 1. For example $\phi(1) = 1$ since the only natural number coprime with 1 is itself (1 is the only positive integer that is coprime with itself), $\phi(6) = 2$ since 1 and 5 are coprime with 6, but 2, 3, and 4 are not. Verify the following special cases of important properties of the Euler totient function.

a) $\phi(17) = 16$ (special case of $\phi(p) = p - 1,\ p$ prime)

b) $\phi(1) + \phi(2) + \phi(4) + \phi(8) = 8$ (special case of $\sum_{k|n} \phi(k) = n$)

c) $\phi(15) = \phi(3)\phi(5)$ (special case of $\phi(mn) = \phi(m)\phi(n)$, m, n relatively prime)

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[2] Keep in mind that injections (or 1-1 functions) do not map different points into the same point and hence a 1-1 map from A to B means the cardinality of A is less than or equal to the cardinality of B.

d) $\phi(5^3) = (5-1)5^2$ special case of $\phi(p^k) = (p-1)p^{k-1}$

Ans: a) The numbers relatively prime to any prime number p are 1, 2, …, $p-1$.

b), c), d) Direct verification. The reader can attempt to prove the general cases of these properties. If unsuccessful the proofs can easily be found online. The totient function has many important properties and plays an important role in number theory, abstract algebra, and even in coding theory.

35. **(Carmichael's Totient Function Conjecture)** An open question in number theory is the Carmichael Totient Function Conjecture states that for every natural number n there is at least one other natural number $m \neq n$ such that $\phi(m) = \phi(n)$. This conjecture was first stated by the American mathematician Robert Carmichael in 1907 and still has not been proven or disproved. Verify that the conjecture is true for natural numbers 6, 8 and 15.
Ans: For $n = 6$ we have $\phi(6) = 2 = \phi(3)$. For $n = 8$ we have $\phi(8) = 4 = \phi(5)$. For $n = 15$ we have $\phi(15) = 8 = \phi(16)$.

ΦΛΜΣΩΖΞ