Section 3.5: Image and Inverse Image of a Set

1**. (Party Time)** We are having a party with possible desserts $B=\{\text{cake, ice cream, pie}\}$ where the possible guests are among the group $A=\{a,b,c,d,e\}$. Each guest's favorite dessert is indicted by the function $f:A\to B$ where

$$f(a)=\text{pie}, f(b)=\text{ ice cream}, f(c)=\text{ pie}, f(d)=\text{ ice cream}, f(e)=\text{ cake}$$

What types of desserts will be required if the following groups of guests ae invited to the party?

a) $f(\{a,c\})$ **Ans:** $f(\{a,c\})=\{\text{pie}\}$

b) $f(\{b,d,e\})$ **Ans:** $f(\{b,d,e\})=\{\text{ice cream, cake}\}$

c) $f(\{b\})$ **Ans:** $f(\{b\})=\{\text{ice cream}\}$

d) $f(\varnothing)$ **Ans:** $f(\varnothing)=\varnothing$

e) $f(\{a,b,c,d,e\})$ **Ans:** $f(\{a,b,c,d,e\})=\{\text{ice cream, cake, pie}\}$

2. **(Images of Sets)** Given the sets $A=\{1,2,3,4\}$, $B=\{a,b,c,d\}$ and the function $f:A\to B$ defined by $f(1)=b,\ f(2)=a,\ f(3)=d,\ f(4)=c$, find the following.

a) $f(\{1,3\})$ **Ans:** $f(\{1,3\})=\{b,d\}$

b) $f(\{2,3,4\})$ **Ans:** $f(\{2,3,4\})=\{a,c,d\}$

c) $f(\{2\})$ **Ans:** $f(\{2\})=\{a\}$

d) $f(\{1,2,3,4\})$ **Ans:** $f(\{1,2,3,4\})=\{a,b,c,d\}$

e) $f^{-1}(\{a,c\})$ **Ans:** $f^{-1}(\{a,c\})=\{2,4\}$

f) $f^{-1}(\{a,b,c\})$ **Ans:** $f^{-1}(\{a,b,c\})=\{1,2,4\}$

3. **(Interpretation of Images)** Translate the following statements. For example $y\in f(A)$ means there exists an $x\in A$ such that $y=f(x)$.

a) $y\in f(A\cup B)$

Ans: $y\in f(A\cup B)$ means $\exists x\in A\cup B$ such that $y=f(x)$

b) $y\in f(A)\cup f(B)$

Ans: $y\in f(A)\cup f(B)$ means

$$(\exists a\in A \text{ such that } y=f(a))\vee(\exists b\in B \text{ such that } y=f(b))$$

c) $y \in f(A \cap B)$

Ans: $y \in f(A \cap B)$ means $\exists x \in A \cap B$ such that $y = f(x)$

d) $y \in f(A) \cap f(B)$

Ans: $y \in f(A) \cap f(B)$ means $(\exists a \in A \Rightarrow y = f(a)) \wedge (\exists b \in B \Rightarrow y = f(b))$

e) $y \in f\left(\bigcup_{i \in I} A_i\right)$

Ans: $y \in f\left(\bigcup_{i \in I} A_i\right)$ means $\exists a \in f(A_i)$ for some A_i, $i \in I$ such that $y = f(a)$.

f) $y \in \bigcup_{i \in I} f(A_i)$

Ans: $y \in \bigcup_{i \in I} f(A_i)$ means $\exists a_i \in f(A_i)$ for some A_i, $i \in I$ such that $y = f(a_i)$.

g) $y \in f\left(\bigcap_{i \in I} A_i\right)$

Ans: $y \in f\left(\bigcap_{i \in I} A_i\right)$ means $\exists a \in A_i, \forall i \in I$ such that $y = f(a)$

h) $y \in \bigcap_{i \in I} f(A_i)$

Ans: $y \in \bigcap_{i \in I} f(A_i)$ means $\exists a_i \in A_i, \forall i \in I$ such that $y = f(a_i)$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 2$. Find the following sets.

a) $f(\{-1,1,3\})$ **Ans:** $f(\{-1,1,3\}) = \{3,10\}$

b) $f(\varnothing)$ **Ans:** $f(\varnothing) = \varnothing$

c) $f([0,2])$ **Ans:** $f([0,2]) = [2,6]$

d) $f([-1,2] \cup [3,5])$ **Ans:** $f([-1,2] \cup [3,5]) = [3,6] \cup [10,26]$

e) $f^{-1}([-1,2])$ **Ans:** $f^{-1}([-1,2]) = [3,6]$

f) $f^{-1}([0,2])$ **Ans:** $f^{-1}([0,2]) = \{0\}$

g) $f^{-1}([5,10])$ **Ans:** $f^{-1}([5,10]) = [-3,-2] \cup [2,3]$

h) $f^{-1}([-1,5])$ **Ans:** $f^{-1}([-1,5]) = [-2,2]$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x| + 1$. Find the following sets.

a) $f([-2,-1))$ **Ans:** $f([-2,-1))=[2,3]$
b) $f([-2,3])$ **Ans:** $f([-2,3])=[1,4]$
c) $f([-2,2])$ **Ans:** $f([-2,2])=[1,3]$
d) $f^{-1}([0,4])$ **Ans:** $f^{-1}([0,4])=[-3,3]$
e) $f^{-1}([-2,0])$ **Ans:** $f^{-1}([-2,0])=\varnothing$
f) $f^{-1}(\{1,2,3\})$ **Ans:** $f^{-1}(\{1,2,3\})=\{0,-1,1,-2,2\}$

6. **(Identity or Falsehood?)** Let $f:A\to B$ and let X,Y be subsets of A. Prove or disprove

$$A\subseteq B \Rightarrow f(A)\subseteq f(B).$$

7. **(Image of a Union)** Show $f(A\cup B)\subseteq f(A)\cup f(B)$.

Ans: $y\in f(A\cup B)\Rightarrow \exists x\in A\cup B$ such that $y=f(x)$ .We have two cases. In case $x\in A$, we have $y=f(x)\in f(A)\subseteq f(A)\cup f(B)$. In case $x\in B$ we have $y=f(x)\in f(B)\subseteq f(A)\cup f(B)$. Hence $y\in f(A)\cup f(B)$.

8. **(Inverse of Union)** Show $f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B)$.
Ans:
$(\subseteq)$

$$\begin{aligned} x\in f^{-1}(A\cup B) &\Rightarrow f(x)\in A\cup B \\ &\Rightarrow f(x)\in A \text{ or} f(x)\in B(x)\in A\cup B \\ &\Rightarrow f(x)\in A \text{ or} f(x)\in B \quad . \quad . \\ &\Rightarrow x\in f^{-1}(A) \text{ or } x\in f^{-1}(B) \\ &\Rightarrow x\in f^{-1}(A)\cup f^{-1}(B) \end{aligned}$$

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$(\supseteq)$ Do the previous steps backwards.

9. **(Compliment Identity)** Show $f^{-1}(\overline{A})=\overline{f^{-1}(A)}$

Ans: $(\subseteq)$ $x\in f^{-1}(\overline{A})\Rightarrow f(x)\in \overline{A}\Rightarrow f(x)\notin A\Rightarrow x\notin f^{-1}(A)\Rightarrow x\in \overline{f^{-1}(A)}$.
$(\supseteq)$ $x\in \overline{f^{-1}(A)}\Rightarrow x\notin f^{-1}(A)\Rightarrow f(x)\notin A\Rightarrow f(x)\in \overline{A}\Rightarrow x\in f^{-1}(\overline{A})$

10. **(Composition of a Function with Its Inverse)** Show the following and give examples to show we do not have to prove equality.

a) $f\left[f^{-1}(A)\right]\subseteq A$

Ans:
$y\in f\left[f^{-1}(A)\right]\Rightarrow \exists x\in f^{-1}(A)\ni y=f(x)\Rightarrow y=f(x)\in A$. To show A is not a subset of $f\left[f^{-1}(A)\right]$, let $f(x)=x^2$ and $A=\left[-2,-1\right]$. In this case $f\left[f^{-1}(A)\right]=\varnothing$ .Hence $A\not\subseteq f\left[f^{-1}(A)\right]$.

b) $f^{-1}\left[f(A)\right]\supseteq A$

Ans: $x\in A\Rightarrow f(x)\in f(A)\Rightarrow x\in f^{-1}\left[f(A)\right]$. To see $f^{-1}\left[f(A)\right]$ is not a subset of A, let $A=\left[0,1\right]$, $f(x)=x^2$ and so $f^{-1}\left(f(A)\right)=f^{-1}\left(\left[0,1\right]\right)=\left[-1,1\right]$. Hence $f^{-1}\left[f(A)\right]\not\subseteq A$.

11. Let $f:\mathbb{N}\to\mathbb{R}$ be a function defined by $f(n)=1/n$. Find

a) $f^{-1}\left(\left[\frac{1}{10},1\right]\right)$ **Ans:** $f^{-1}\left(\left[\frac{1}{10},1\right]\right)=\{1,2,3,...,9,10\}$

b) $f^{-1}\left(\left[\frac{1}{100},\frac{1}{2}\right]\right)$ **Ans:** $f^{-1}\left(\left[\frac{1}{100},\frac{1}{2}\right]\right)=\{2,3,...,99,100\}$

c) $f^{-1}\left(\left[0,\frac{1}{10}\right]\right)$ **Ans:** $f^{-1}\left(\left[0,\frac{1}{10}\right]\right)=\{n\in\mathbb{N}:n\geq 10\}$

12. **(Inverse Image of an Open Interval)** In topology, a continuous function f is defined as function if the inverse image of every open set in the range is an open set in the domain. Show that for the continuous function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2$, the inverse image of the following open intervals[1] is an open interval or the union of open intervals.

a) $f^{-1}\left((-1,1)\right)$ **Ans:** $f^{-1}\left((-1,1)\right)=(-1,1)$ open

b) $f^{-1}\left((0,4)\right)$ **Ans:** $f^{-1}\left((0,4)\right)=(-2,0)\cup(0,2)$ open

c) $f^{-1}(\mathbb{R})$ **Ans:** $f^{-1}(\mathbb{R})=\mathbb{R}$ open

d) $f^{-1}\left((4,16)\right)$ **Ans:** $f^{-1}\left((4,16)\right)=(-2,-4)\cup(2,4)$ open

[1] Open intervals and union of open intervals are special cases of open sets and the real number system is a special topological space.

13. **(Dirichlet's Function)** Dirichlet's function[2] $f:[0,1]\to R$ is defined by by

$$f(x)=\begin{cases}1 & x \text{ is rationial}\\ 0 & x \text{ is irrational}\end{cases} \qquad 0\le x\le 1$$

Find

a) $f^{-1}\left(\left[\frac{1}{2},1\right]\right)$ **Ans:** $f^{-1}\left(\left[\frac{1}{2},1\right]\right)=\{x\in\mathbb{R}: x \text{ rational}\}$

b) $f^{-1}\left(\left[0,\frac{1}{2}\right]\right)$ **Ans:** $f^{-1}\left(\left[0,\frac{1}{2}\right]\right)=\{x\in\mathbb{R}: x \text{ irrational}\}$

14. Let $f:X\to Y$. Show that for $x\in X$ one has $f(\{x\})=\{f(x)\}$.

Ans: The set $f(\{x\})\subseteq Y$ consists of all images of $\{x\}\subseteq X$. But there is only one element $x\in\{x\}$, so the set of images is simply $\{f(x)\}$.

15. **(Connected Sets)** It can be proven that the continuous image of a connected set is connected[3]. Show that the image of the connected set $[-1,1]$ under the functions.

a) $f(x)=x^3$ **Ans:** $f([-1,1])=[-1,1]$

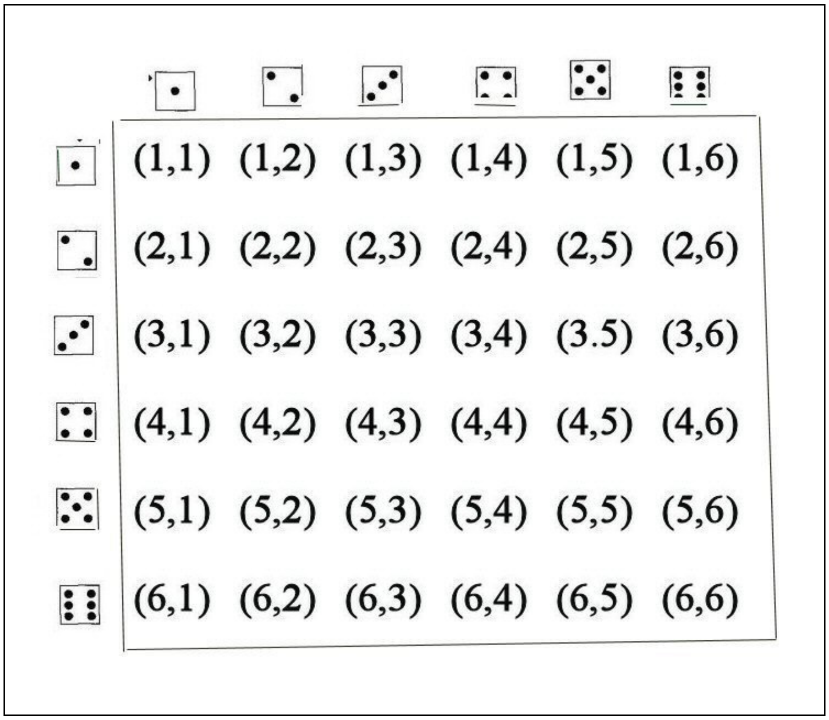
b) $f(x)=e^x$ **Ans:** $f([-1,1])=[e^{-1},e]$

c) $f(x)=2x+1$ **Ans:** $f([-1,1])=[-1,3]$

16. **(Finite Probability)** Mappings between sets is more common that you think. The diagram below shows the 36 possible equally likely outcomes when rolling a pair of dice.

[2] Sometimes called the "shotgun" function since it is full of holes

[3] Connectness is a precise topological concept we won't go into here. Use your intuition of what it might mean for a set to be connected.



(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3.5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The probability that any collection of outcomes occurs (like "rolling" a 7) is a function $f: P(A) \to [0,1]$ from the power set of A to the closed interval $[0,1]$. Find the following images.

a) $f\left(\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}\right)$ a match

Ans: 6/36 = 1/6

b) $f\left(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}\right)$ rolling a 7

Ans: 6/36 = 1/6

c) $f\left(\{(1,1)\}\right)$ snake eyes

Ans: 1/36

d) $f\left(\{(1,3),(2,2),(3,1)\}\right)$ rolling a 4

Ans: 3/36

17. **(Function of Functions[4])** Define $C[0,1]$ to be the set of all continuous functions defined on the unit interval $[0,1]$. For example, the function defined by $f(x)=x^2$ would belong to $C[0,1]$. Now, define a function $I: C[0,1] \to \mathbb{R}$ by $I(f)=\int_0^1 f(x)\,dx$.

[4] A function of a function is called a functional.

a) Find the functions $f, g \in C[0,1]$ so $I(f)=1,\ I(g)=0.5,\ I(h)=-4$

Ans: $f(x)=1,\ g(x)=x,\ h(x)=-4$

b) Express the integral property

$$\int_0^1 [f(x)+g(x)]dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$$

in terms of the function I.

.

c) If the function I 1-1 ?

Ans: no, $f(x)=x,\ g(x)=1-x$ both have $I(f)=I(g)=1/2$.

d) Is the function onto $\mathbb{R}$?

Ans: The function I is onto. For each $a \in \mathbb{R}$, there exists an $f \in C[0,1]$ that maps into a, namely the constant function $f(x)=a$, whose image is $I(f)=\int_0^1 f(x)dx = \int_0^1 a\,dx = a$.

18. **(Hmmmmmmmmmm)** If $f : X \to Y$, the set function $f(X)$ can be thought of a new function $F : P(X) \to P(Y)$ from the power set of X to the power set of Y. If the function f is 1-1, then is the function F 1-1? In other words, if f maps different points $x \in X$ into different points $y \in Y$, does F map different subsets of X into different subsets of Y ?

Ans: The answer is yes.

Let $A \neq B$ where $A, B \in P(X)$. Hence there exists an element in one set that is not in the other set. Suppose $a \in A - B$. Since f is 1-1 we have $f(a) \in f(A) - f(B)$ which means $F(A) \neq F(B)$ which means F is 1-1.

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