Section 4.1 Construction of the Real Numbers

1. **(Equivalence Relation I)** Show that the relation $\equiv$ defined by

$$(m,n)\equiv(m',n') \text{ if and only if } m+n'=m'+n$$

between pairs of of *natural numbers* (m,n) and (m',n') is an equivalence relation.

Ans: **reflexive**: $(m,n)\equiv(m,n)$ since $m+n=m+n$. Hence $\equiv$ is reflexive.
symmetric: If $(m,n)\equiv(m',n')\Leftrightarrow m+n'=m'+n \Leftrightarrow (m',n')\equiv(m,n)$
transitive: Let $(m,n)\equiv(m',n')$ and $(m',n')\equiv(m'',n'')$**.** Hence

$m+n'=m'+n,\ m'+n''=m''+n'$. Adding these two equations and subtracting the common factors we get $m+n''=m''+n$ which says $(m,n)\equiv(m'',n'')$. Hence, the relation is transitive.

2. **(Equivalence Relation II)** Show that the relation $\equiv$ defined by

$$(m,n)\equiv(m',n') \Leftrightarrow mn'=m'n$$

between pairs of *integers* (m,n) and (m',n') is an equivalence relation.

Ans: **reflexive**: $(m,n)\equiv(m,n)$ since $mn=mn$. Hence $\equiv$ is reflexive.
symmetric: If $(m,n)\equiv(m',n')\Leftrightarrow mn'=m'n \Leftrightarrow (m',n')\equiv(m,n)$
transitive: Let $(m,n)\equiv(m',n')$ and $(m',n')\equiv(m'',n'')$**.** Hence

$mn'=m'n,\ m'n''=m''n'$. Multiplying these equations and canceling the common factor $m'n'$ we get $mn'm'n''=m'nm''n'\Rightarrow mn''=m''n$. Hence $(m,n)\equiv(m'',n'')$ and so the relation is transitive.

3. **(Arithmetic in $\mathbb{Z}$)** We have created the integers $\mathbb{Z}$ from equivalence classes of natural numbers pair (m,n). When we see $(3,5)$ we think "-2", when we see $(6,3)$ we think "3" and so on. Perform the following arithmetic steps on pairs of natural numbers.

a) $\overline{(1,5)}\oplus\overline{(3,2)}$ **Ans:** $\overline{(1,5)}\oplus\overline{(3,2)}=\overline{(4,7)}$ or -3

b) $\overline{(1,5)} \ominus \overline{(3,2)}$ **Ans:** $\overline{(1,5)} \ominus \overline{(3,2)}=\overline{(3,8)}$ or -5

c) $\overline{(1,5)}\otimes\overline{(3,2)}$ **Ans:** $\overline{(1,5)}\otimes\overline{(3,2)}=\overline{(13,17)}$ or -4

4. **(Arithmetic in $\mathbb{Q}$)** $\mathbb{Z}$) We have created the rational numbers $\mathbb{Q}$ from equivalence classes of pairs of integers (m,n). When we see $(3,5)$ we think "3/5", when we see $(6,3)$ we think "6/3 = 2" and so on. Perform the following arithmetic steps on pairs of integers.

a) $\overline{(1,5)} \oplus \overline{(3,2)}$ **Ans:** $\overline{(1,5)} \oplus \overline{(3,2)} = \overline{(17,10)}$ or $\dfrac{17}{10}$

b) $\overline{(1,2)} \ominus \overline{(3,2)}$ **Ans:** $\overline{(1,2)} \ominus \overline{(4,9)} = \overline{(1,18)}$ or $\dfrac{1}{18}$

c) $\overline{(1,5)} \otimes \overline{(3,2)}$ **Ans:** $\overline{(1,5)} \otimes \overline{(3,2)} = \overline{(3,10)}$ or $\dfrac{3}{10}$

5. **(Decimal to Fractions)** Find the fraction for each of the following numbers in decimal form.

a) 0.9999…. $(0.\overline{9})$

Ans:

$$\begin{aligned}0.9999\ldots &= 0.9+0.09+.009+\cdots\\ &= \frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots\\ &= \frac{9}{10}\left(1+\frac{1}{10}+\frac{1}{100}+\cdots\right)\\ &= \frac{9}{10}\left(\frac{1}{1-(1/10)}\right)\\ &= \frac{9}{10}\left(\frac{10}{9}\right)\\ &= 1\end{aligned}$$

b) 0.23232323…. $(0.\overline{23})$

Ans:

$$\begin{aligned}0.2323\ldots\ldots &= 0.23+0.0023+.000023+\cdots\\ &= \frac{23}{100}+\frac{23}{10000}+\cdots\\ &= \frac{23}{100}\left(1+\frac{1}{100}+\frac{1}{10000}+\cdots\right)\\ &= \frac{23}{100}\left(\frac{1}{1-(1/100)}\right)\\ &= \frac{23}{100}\left(\frac{100}{99}\right)\\ &= \frac{23}{99}\end{aligned}$$

c) 0.0123123123… $(0.0\overline{123})$

Ans:

$$\begin{aligned} 0.0125125\cdots &= \frac{125}{1000}\sum_{k=0}^{\infty}(0.001)^k \\ &= \frac{125}{1000}\left(\frac{1}{1-0.001}\right) \\ &= \frac{125}{9990} \end{aligned}$$

d) 0.001111… $\left(0.00\bar{1}\right)$

Ans:

$$\begin{aligned} 0.0011111\cdots &= \frac{1}{1000}\left(1+\frac{1}{10}+\frac{1}{100}+\cdots\right) \\ &= \frac{1}{1000}\left(\frac{1}{1-(1/10)}\right) \\ &= \frac{1}{1000}\left(\frac{10}{9}\right) \\ &= \frac{1}{900} \end{aligned}$$

In general the fraction resulting from a rational number in decimal form is

$$0.\underbrace{000000}_{n \text{ zeros}}\underbrace{d_1 d_2 \ldots d_k}_{k \text{ repeats}}\underbrace{d_1 d_2 \ldots d_k}_{k \text{ repeats}}\cdots = \frac{d_1 d_2 \ldots d_k}{\underbrace{999\ldots999}_{k \text{ nines}}\underbrace{000\ldots0}_{n \text{ zeros}}}$$

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