Section 4.2 The Complete Ordered Field: The Real Numbers

Normally when we solve elementary algebra problems the rules of the game are so entrenched in the back of our minds that we use them without much thought. Solve Problems 1-4 using the axioms of an algebraic field and tell which axioms you are using.

1. **(Parenthesis Not Needed in Addition)** Show that in the sum of four elements $a,b,c,d \in \mathbb{R}$ the parenthesis is not needed. Hint: Use the associative rule for addition and show that no matter how parenthesis are inserted in the sum, say like $\left(a+\left(b+c\right)\right)+d$ the parenthesis can always be moved" to the left like $\left(\left(a+b\right)+c\right)+d$.

Ans: For each of the 7 ways one can insert one or two parenthesis between the terms of $a+b+c+d$ one can use the associative law move the parenthesis to the left, getting $\left(\left(a+b\right)+c\right)+d$. Since they are all the same, we can simply write $a+b+c+d$.

2. **(Parenthesis Not Needed in Multiplication)** Show that in the product of four elements $a,b,c,d \in \mathbb{R}$ the parenthesis is not needed. Hint: Use the associative rule for multiplication and show that no matter how parenthesis are inserted in the product, say like $\left(a\left(bc\right)\right)d$ the parenthesis can always be "moved" to the left like $\left(\left(ab\right)c\right)d$.

Ans: For each of the 7 ways one can insert one or two parenthesis between the factors of $abcd$ one can use the associative law move the parenthesis to the left, getting $\left(\left(ab\right)c\right)d$. Since they are all the same, we can simply write $abcd$.

3. **(Solving a Middle School Equation)** Show that $\left(\forall a,b \in \mathbb{R}\right)$ the equation $a+x=b$ has exactly one solution, given by solution $x=b+\left(-a\right)$.

Ans: Suppose $x \in \mathbb{R}$ satisfies $a+x=b$. Then

$$x = x+0 = x+\left(a+\left(-a\right)\right) = \left(x+a\right)+\left(-a\right) = b+\left(-a\right)$$

and so $x=b+\left(-a\right)$ is the only possible solution. Plugging this candidate into the given equation we get .

$$a+x = a+\left(b+\left(-a\right)\right) = a+\underbrace{\left(\left(-a\right)+b\right)}_{\text{commutitive law}} = \underbrace{\left(a+\left(-a\right)\right)+b}_{\text{associative law}} = 0+b = b$$

4. **(A Negative of a Negative …)** Show that $\left(\forall a \in \mathbb{R}\right)\left[a=-\left(-a\right)\right]$
Ans: Since from Problem 4 we know there is only one solution of the equation $a+x=b$, and we now show both a and $-\left(-a\right)$ are solutions of $x+\left(-a\right)=0$ so they

must be the same. We saw in Problem 3 $x = a$ is a solution, but we also know $-(-a)+(-a)=0$ from the field axiom A3. Hence $a = -(-a)$.

5. **True or False**

a) The natural numbers $\mathbb{N}$ is an ordered field using the usual operations of addition and multiplication.

Ans: False, numbers don't have negative inverses among other things.

b) $\mathbb{Q}$ and $\mathbb{R}$ are both ordered fields but $\mathbb{C}$ is not.
Ans: True

c) For A, B bounded sets, $\sup(A-B) = \sup(A) - \sup(B)$ where we define

$$A - B = \{a - b : a \in A,\ b \in B\}.$$

Ans: False, let $A = \{1,2\}, B = \{4,5\}$ then $A - B = \{-3,-4,-2\}$.Hence

$$\sup(A-B) = -2,\ \ \sup(A) - \sup(B) = 2 - 5 = -3.$$

d) The integers $\mathbb{Z}$ form an ordered field.
Ans: False, not all elements have a multiplicative inverse. The integer 3 has no multiplicative inverse (1/3 is not an integer). The integers are said to form an algebraic ring, not a field.

e) All finite sets of real numbers have a least upper bound.
Ans: True, they are bounded so they have a least upper bound.

f) The set of rational numbers less than 1 has a supremum.
Ans: True, the sup is 1.

g) If a subset of the real numbers has an upper bound, then it has exactly one least upper bound.
Ans: True, can't argue with an axiom. That's the completeness axiom.

h) $\sup(\mathbb{Z}) = \infty$
Ans: False, the integers are not bounded above so there is no sup.

i) Every finite set can be ordered.
Ans: True, simply take members at random and order them in the order they are chosen.

j) The set of linear functions $f(x) = ax + b$ with addition and multiplication of functions defined in the usual way is an algebraic field.
Ans: False, not all functions have a multiplicative inverse.

k) When it comes right down to it the completeness axiom ensures there are no "holes" in the real numbers.
Ans: True, that's about it.

6. For the following sets A find (if they exist), $\max(A), \min(A)$, $\sup(A), \inf(A)$.

a) $A = \{1,3,9,4,0\}$

Ans: $\max(A) = 9$, $\min(A) = 0$, $\sup(A) = 9$, $\inf(A) = 0$

b) $A = [0, \infty)$

Ans no max, $\min(A) = 0$, no sup, $\inf(A) = 0$

c) $A = \{x \in \mathbb{Q} : 0 \le x < 1\}$

Ans no max, $\min(A) = 0$, $\sup(A) = 1$, $\inf(A) = 0$

d) $A = [-1, 3]$

Ans $\max(A) = 3$, $\min(A) = -1$, $\sup(A) = 3$, $\inf(A) = -1$

e) $A = \{x : x^2 - 1 = 0\}$

Ans no max, $\min(A) = -1$, no sup, $\inf(A) = -1$

f) $A = \{n \in \mathbb{N} : n \text{ divides } 100\}$

Ans Student Project

g) $A = \{x \in \mathbb{R} : x^2 < 2\}$

Ans no max, no min, $\sup(A) = \sqrt{2}$, $\inf(A) = -\sqrt{2}$

h) $A = (-\infty, \infty)$

Ans no max, no min, no sup, no inf

i) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$

Ans $\max(A) = 1$, no min, $\sup(A) = 1$, $\inf(A) = 0$

7. (**More Difficult Sup and Inf**) If they exist, find the supremum and infimum of the set

$$A = \left\{\frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N}\right\}.$$

Ans: The set is bounded between -1 and 1 since

$$0 < \frac{1}{n} \le 1,\ 0 < \frac{1}{m} \le 1 \Rightarrow -1 = 0 - 1 < \frac{1}{n} - \frac{1}{m} < 1 + 0 = 1$$

Hence, we suspect $\sup(A) = 1,\ \inf(A) = -1$. To show $\sup(A) = 1$ suppose it is false. That is, there is a smaller upper bound u smaller than 1. But this gives a contradiction since if we pick $n = 1$ and let $m \to \infty$ the sequence

$$\frac{1}{n} - \frac{1}{m}$$

will eventually get larger than any $u < 1$. Hence, $\sup(A) = 1$.We can show $\inf(A) = -1$ by assuming there is a larger lower bound, say $-1 < v$. But this also leads to a contradiction since we can pick $m = 1$ and by letting $n \to \infty$ the sequence

$$\frac{1}{n} - \frac{1}{m}$$

will eventually get smaller than any $-1 < v$. Hence, $\inf(A) = -1$.

8.**(Algebraic Field)** Show that the rational numbers with the operations of addition and multiplication form a algebraic field.
Ans: The reader can check that this system satisfies the axioms of a field.

9. **(Boolean Field)** Show that the set $F_2 = \{0,1\}$ consisting of 2 elements forms an algebraic field.
Ans: The reader can check that this set with the addition and multiplication described in the text satisfies the axioms of a field.

10. **(Ordered Field)** Show that the rational numbers with the operations of addition and multiplication and the usual “less than” order relation $"<"$ forms an ordered field.
Ans: The reader can check that this set with the addition and multiplication described in the text satisfies the axioms of a field.

11. **(Not an Ordered Field)** Show that the field $\mathbb{C}$ of complex numbers is not an ordered field.
Ans: First assume $i = \sqrt{-1} > 0$. Hence, $i^2 = -1 > 0$ and adding 1 gives $0 > 1$. But squaring $-1 > 0$ gives $1 > 0$ and so we have proven both $0 > 1$ and $0 < 1$ which contradicts anti-symmetry axiom for the order relation. A similar contradiction is reached if we assume $i < 0$. Hence, we cannot order the complex numbers.

12. **(Well-Ordering Principle)** The **well-ordering principle**[1] states that every (non-empty) subset $A \subseteq \mathbb{N}$ contains a smallest element under the usual ordering $\leq$. Does this principle hold for subsets $A \subseteq \mathbb{Z}$?
Ans: No, the integers do not have a smallest member (nor do many subsets of the integers)

13. **(Well-Ordering Theorem)** A partial order $"\preceq"$ on a set X is called a **well ordering** (and the set X is called **well ordered**) if every nonempty subset $S \subseteq X$ has a least element (i.e. belongs to S). The **Well Ordered Theorem**[2] states that every set can be well ordered by some partial order (i.e. there exists a well ordering $"\preceq"$ on X). Are the following sets well ordered by the usual "less than or equal to" order $"\leq"$?

a) $\mathbb{N}$ **Ans:** yes
b) $\{3,4,5\}$ **Ans:** yes
c) $\mathbb{Z}$ **Ans:** no
d) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ **Ans:** no

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[1] This principle is really a theorem and is equivalent to the Principle of Mathematical Induction.
[2] The Well Ordering Theorem is equivalent to the Axiom of Choice and was proven by the German mathematician Ernst Zermelo (1871-1953). Although the theorem says the real numbers $\mathbb{R}$ are well ordered, no one has ever found a well ordering.