Section 4.3 : Complex Numbers

1. For the following complex numbers, find $|z|$, $\theta = \arg(z)$, $w+z$, wz, w/z and plot the locations of $w, z, w+z, wz,$ and w/z in the complex plane.

a) $w = i,\ z = 1+i$

Ans: $|z| = \sqrt{2},\ \theta = \pi/4,\ w+z = 1+2i,\ wz = -1+i,\ w/z = (1+i)/2$

b) $w = 2,\ z = -i$

Ans: $|z| = 1,\ \theta = 3\pi/2,\ w+z = 2-i,\ wz = -2i,\ w/z = 2i$

c) $w = i,\ z = -i$

Ans $|z| = 1,\ \theta = 3\pi/2,\ w+z = 0,\ wz = 1,\ w/z = -1$

d) $w = -1-i,\ z = 1+i$

Ans: $|z| = \sqrt{2},\ \theta = \pi/4,\ w+z = 0,\ wz = -2i,\ w/z = -1$

e) $w = 1+i,\ z = -1+i$

Ans: $|z| = \sqrt{2},\ \theta = 3\pi/4,\ w+z = 2i,\ wz = -2,\ w/z = -i$

2. **(Convert to Polar Form)** Convert the following complex numbers to polar form $re^{i\theta}$.

a) $2i$ **Ans:** $re^{i\theta} = 2e^{i(\pi/2)2}$

b) $-1+i$ **Ans:** $re^{i\theta} = \sqrt{2}e^{i(3\pi/4)}$

c) $-i$ **Ans:** $re^{i\theta} = e^{i(3\pi/2)}$

d) $2-2i$ **Ans:** $re^{i\theta} = \sqrt{8}e^{i(7\pi/8)}$

e) 3 **Ans:** $re^{i\theta} = 3e^{i(0)}$

3. **(Convert to Cartesian Form)** Convert the following complex number to cartesian form $x+iy$.

a) $e^{3\pi i}$

Ans: $e^{3\pi i} = \cos 3\pi + i\sin 3\pi = -1$

b) $2e^{i\pi/4}$

Ans: $2e^{i\pi/4} = 2[\cos(\pi/4) + i\sin(\pi/4)] = 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

c) $e^{2\pi i}$

Ans: $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$

d) $e^{5\pi i}$

Ans: $e^{5\pi i} = \cos 5\pi + i \sin 5\pi = -1$

e) $5e^{3\pi i/4}$

Ans: $5e^{3\pi i/4} = 5\left[\cos(3\pi/4) + i\sin(3\pi/4)\right] = 5\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$

Margin Note: There used to be a company that sold T-shirts with "Mathematicians, we're Number $e^{2\pi i}$ written on them.

4. Evaluate $(1+i)^{100}$.

Ans:

$$\begin{aligned}(1+i)^{100} &= \left(\sqrt{2}\, e^{i\pi/4}\right)^{100} \\ &= 2^{50}\ e^{25\pi i} \\ &= 2^{50}\left[\cos(25\pi) + i\sin(25\pi)\right] \\ &= 2^{50}(-1) \\ &= -2^{50}\end{aligned}$$

5. Show that the complex conjugate of the sum of two complex numbers is the sum of the conjugates; that is $\overline{(w+z)} = \overline{w} + \overline{z}$.

Ans: Student Project

6. Verify the identity $|z|^2 = z\overline{z}$ for $z = 2+3i$.

Ans: Student Project

7. Find the real and imaginary parts of

a) z^3 **Ans:** $\mathrm{Re}\left(z^3\right) = x^3 - 3xy^2, \mathrm{Im}\left(z^3\right) = 3x^2y - y^3$

b) $1/z$ **Ans:** $\mathrm{Re}(1/z) = \frac{x}{x^2+y^2}, \mathrm{Im}(1/z) = -\frac{y}{x^2+y^2}$

8. Compute

a) $\sqrt{i}$

b) $\sqrt{-i}$

c) $\sqrt{1+i}$

d) $\sqrt[3]{-1}$

e) $\sqrt[4]{-1}$

9. **(de Moivre's Formula)** Use Euler's theorem to prove **de Moivre's formula**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for any positive integer n. Hint: Use induction.

Ans: The formula is clearly true when $n=1$. Assuming the formula is true for n, we can write

$$\begin{aligned}(\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^n \\ &= (\cos\theta + i\sin\theta)(\cos n\theta + i\sin n\theta) \\ &= (\cos\theta\cos n\theta - \sin\theta\sin n\theta) + i(\cos\theta\sin n\theta + \sin\theta\cos n\theta) \\ &= \cos(n+1)\theta + i\sin(n+1)\theta\end{aligned}$$

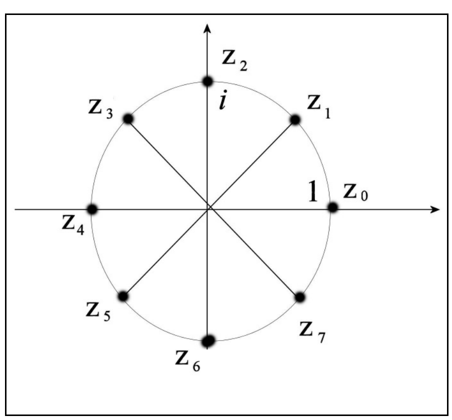
10**. (Primitive Roots of Unity)** The n *roots of unity* are the n roots of the equation $z^n = 1$. Find and plot the roots when $n = 1,2,3,4,$ and 8.
Ans: Writing both sides of the equation $z^n = 1$ in polar form and using de Moivre's Formula, we have

$$r^n(\cos n\theta + i\sin n\theta) = 1\cdot[\cos(0) + i\sin(0)]$$

But two complex numbers are equal if and only if their magnitudes are equal and their angles are equal, hence $r=1$ and the angles $n\theta$ and 0 are equal up to the aiddition of a multiple of 2π. Hence, this gives us $\theta = 0 + 2k\pi$ where k can be any of the numbers $0,1,2,\ldots$. In principle this gives us an infinite number of roots, one for each k, but not all the answers are different since changing the angle by 2π does not change z Hence, we are left with the roots

$$z_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right),\ k = 0,1,2,\ldots,n-1\ .$$

These n complex numbers are scattered equally around the unit circle in the complex plane. The following figure shows the 8 roots of unity.



Eight Roots of Unity

11. **(Fractional Powers)** Find the following.

a) $i^{3/2}$

Ans:

$$\begin{aligned} i^{3/2} &= e^{(3/2)\ln i} \\ &= e^{(3/2)(i\pi/2)} \\ &= e^{3\pi i/4} \\ &= \cos(3\pi/4) + i\sin(3\pi/4) \\ &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \end{aligned}$$

b) $(-1)^{3/4}$

Ans:

$$\begin{aligned} (-1)^{3/4} &= e^{(3/4)(i\pi)} \\ &= e^{(3/4)(i\pi)} \\ &= e^{3\pi i/4} \\ &= \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \end{aligned}$$

c) $\sqrt{1+i}$

Ans:

$$\begin{aligned} \sqrt{1+i} &= (1+i)^{1/2} \\ &= e^{(1/2)\ln(1+i)} \\ &= e^{(1/2)\left(\ln\sqrt{2} + i\pi/4\right)} \\ &= e^{(1/2)\ln\sqrt{2}}\, e^{i\pi/8} \\ &= 2^{1/4}\left(\cos(\pi/8) + i\sin(\pi/8)\right) \\ &\doteq 1.098 + 0.455i \end{aligned}$$

d) i^i

Ans:

$$\begin{aligned} i^i &= e^{i\ln(i)} \\ &= e^{i(i\pi/2)} \\ &= e^{-\pi/2} \\ &\doteq 0.208 \end{aligned}$$

12. **(Hmmmmmmmmm)** Show

$$\frac{1}{i} = -i\ .$$

Ans: Multiply numerator and denominator by i .

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