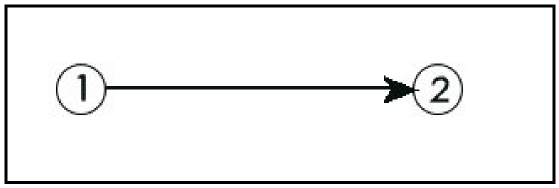
Section 5.2 Directed Graphs

For the digraphs in Problems 1-6, find the adjacency matrix M. Then compute M^2 and $M + M^2$ and verify that the elements of these matrices agree with the number of dominations in the graphs.

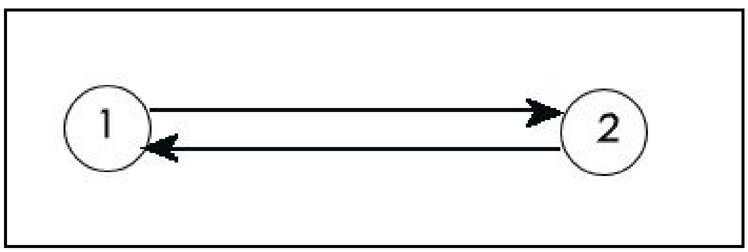
1.



Ans:

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, M + M^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

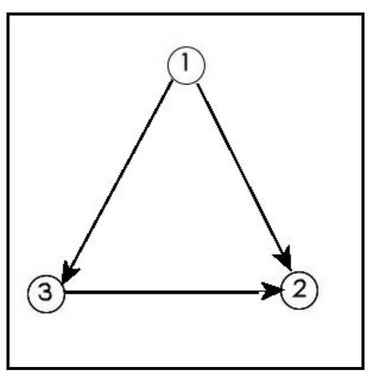
2.



Ans:

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, M^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M + M^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

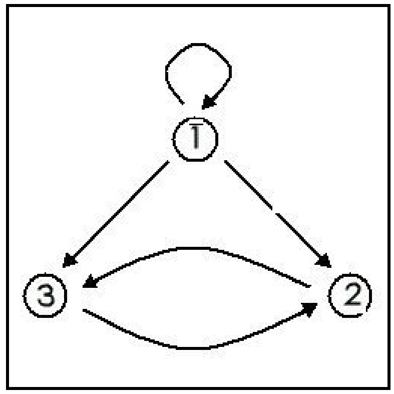
3.



Ans:

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, M^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, M + M^2 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

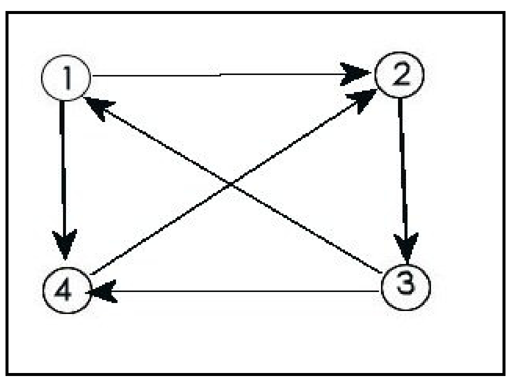
4.



Ans:

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, M^2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M + M^2 = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

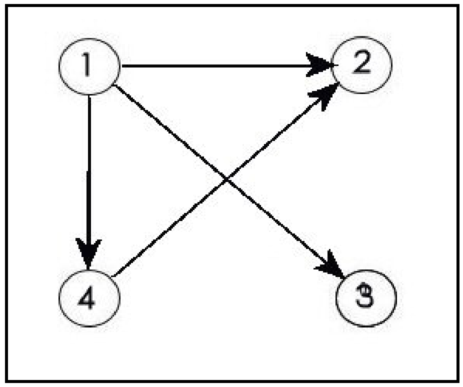
5.



Ans:

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, M^2 = \begin{bmatrix} 0 & 1 & 3 & 0 \\ 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}, M + M^2 = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 3 & 0 & 3 & 3 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

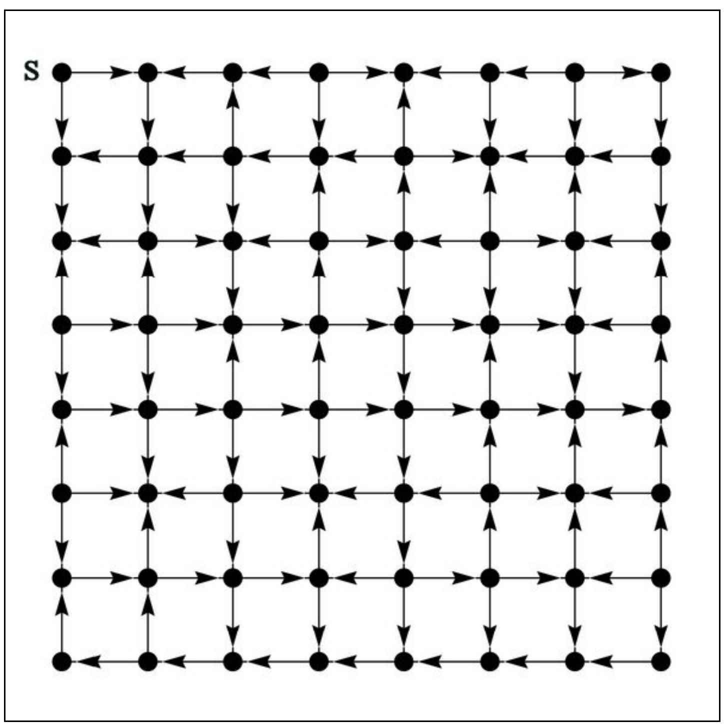
6.



Ans:

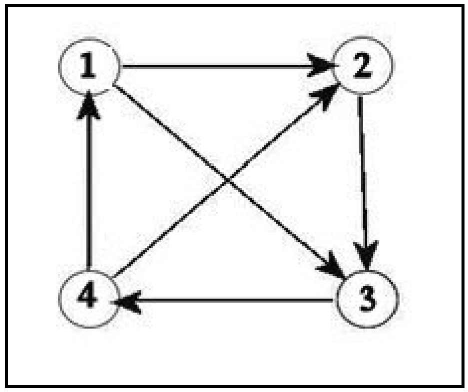
$$M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, M^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M + M^2 = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

7. **(Fun Problem)** Find all nodes that are reachable from the starting node (S) along a directed path.



Ans: In row 1 the nodes in columns 1-5, in row 2 the nodes in columns 2-5, and in row 3 the node in column 2.

8. **Group Dominance** The graph shown in the figure below shows the dominance of a group of four classmates: Andy (1), Betty (2), Charlie (3), and Denise (4).



a) Construct the adjacency matrix M for this graph.
b) Is there a first-stage dominance leader?
c) Compute M^2 and interpret its elements.
d) Who is the group leader?
e) Which person is dominated by the most other people?

Ans: The consensus leader

a)

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

b) Person 1 (Andy) and person 4 (Denise) both dominate 2 after the first round.

c) $M^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ The matrix M^2 says Denise

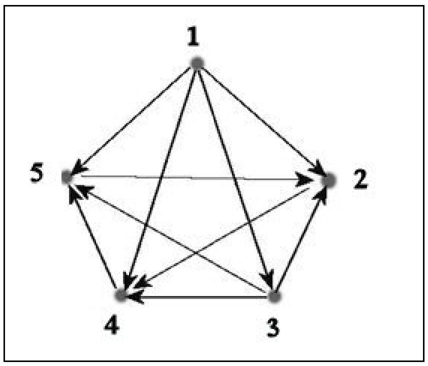
has 3 second-stage dominances which is the largest.

d) $M + M^2 = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}$ Adding first- and second-order dominances we see Denise has 5 which means she is the group leader.

e) Person 2 (Betty) is dominated by 3 people in the first-order and 5 people in first- and second-order.

9. **Round-Robin Tournaments**

The graph shown in the figure below shows the results of a round-robin tournament for five baseball teams.



Round-robin tournament graph

a) Construct the adjacency matrix M for this graph.
b) Is there a consensus leader for the group?
c) Compute M^2 and interpret its elements.
d) Which team is the conference winner?

Ans:

a) $M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

b) Yes, team 1.

c) $M^2 = \begin{bmatrix} 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

e) Still team 1 which has 6 first- and second-order dominances.

10. **Landau's Theorem**

A theorem by Landau states that if some vertex u in a dominance graph has a larger out-degree than all other vertices, then either u dominates all other vertices v, or if it does *not* dominate a given vertex v, then u dominates a third vertex w which in turn dominates v. What does the theorem say in the language of round-robin tournaments? Verify this theorem fro the dominance graphs in Problems 1-6.

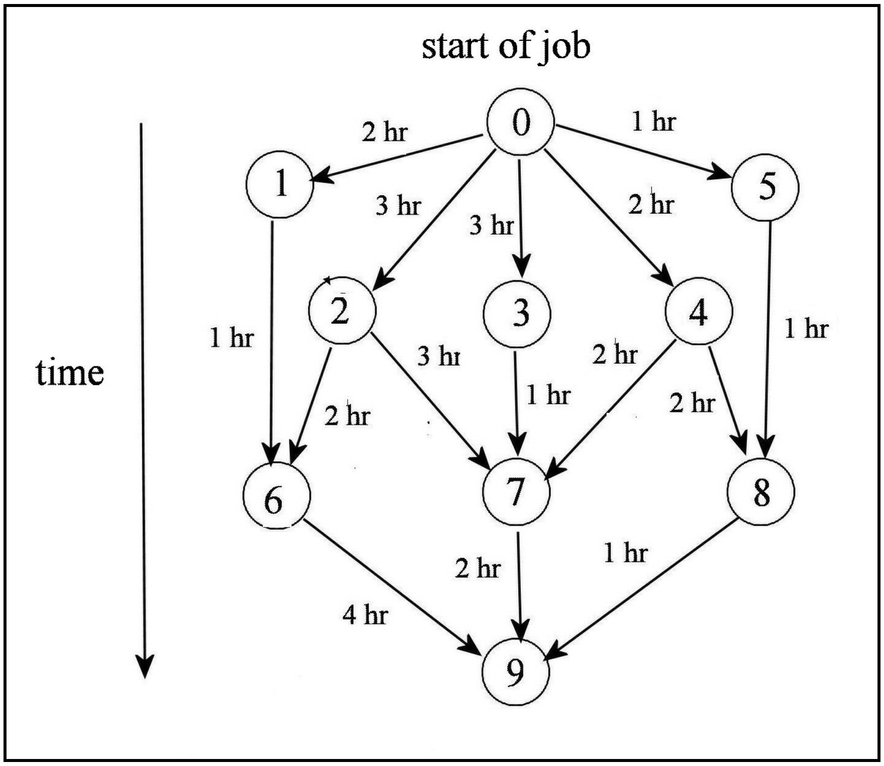
Ans: Landau's theorem says that if a team u wins more games than all other teams then it either beats any other team, say v,or else it beat a third teams, say w that beat team v. For instance, suppose the Southeast Conference plays a round robin football schedule and that Florida State won more games than any other team. Then either Florida State beat every other team, let's say Georgia, or else it beat a third team that beat Georgia.

11. **Landau's Theorem in the Yankee Conference**

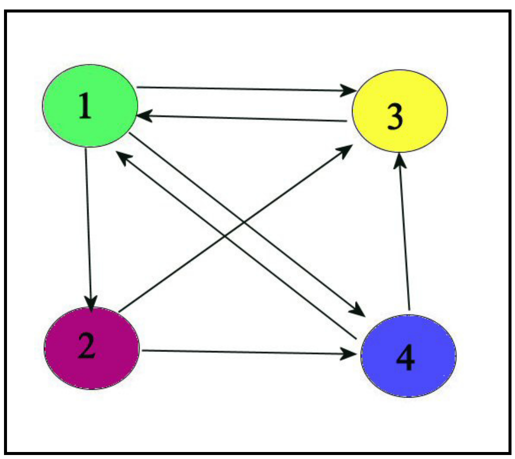
Suppose the football teams in the Yankee Conference play every other team exactly once during the season. At the end of the season Maine has won more games than any other team. However, Maine lost to Vermont. What does Landau's theorem say in the language of the Yankee Conference?

Ans: It says that Maine beat a third team, say New Hampshire, that beat Vermont.

12. **Dynamic Programming** Use dynamic programming to find the shortest way to accomplish the project in the following figure.



13. **(Ranking Webpages)** The following graph illustrates a tiny internet of four webpages where the nodes represent the webpages and the directed edges represent links from one webpage to another. Rank the webpages from first to last.



Ans: The link matrix (or transition matrix as we say in Markov chain theory) is

$$G = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

We now go to our matrix multiplication program or just Google the phrase "Matrix Multiplication Applet" and you will find several online programs to multiply matrices. You can also Google Markov Chain applet and the entire problem can be done online. So doing one of these options we begin with 25% of surfers in each webpage and compute

$$v_1 = Gv_0 = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.2497 \\ 0.2500 \\ 0.2500 \\ 0.2500 \end{bmatrix}$$

Continuing this process, we find

$$v_2 = \begin{bmatrix} 0.375 \\ 0.083 \\ 0.333 \\ 0.208 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0.437 \\ 0.125 \\ 0.271 \\ 0.166 \end{bmatrix}, \ \ldots, \ v_{100} = \begin{bmatrix} 0.381 \\ 0.122 \\ 0.293 \\ 0.194 \end{bmatrix}$$

In other words, after many iterations, we find

38% in webpage 1 (PageRank = 0.38)
12% in webpage 2 (PageRank = 0.12)
29% in webpage 3 (PageRank = 0.29)
19% in webpage 4 (PageRank = 0.15)

ΝΠΧΑΞΑΩ