Section 5.4 Point Set Topology on the Real Line

1. Tell if the following sets subsets of $\mathbb{R}$ are open, closed, both, or neither.

a) $(-1,0)\cup(0,1)$ **Ans:** open

b) $[0,\infty)$ **Ans:** closed

c) $(0,\infty)$ **Ans:** open

d) $\mathbb{N}$ **Ans:** closed

e) $\mathbb{Z}$ **Ans:** closed

f) $\mathbb{Q}$ **Ans:** not open, not closed

g) $A=\{0,1,2,...,100\}$ **Ans:** closed

h) $\{x:|x-1|\}>3$ **Ans:** open

i) $\varnothing$ **Ans:** open and closed

j) $\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4}...\right\}$ **Ans:** not open, not closed

k) $\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4}...\right\}\cup\{0\}$ **Ans:** closed

l) $\{x:x^2>0\}$ **Ans:** open

m) $\bigcup_{n=1}^{\infty}\left(\frac{1}{n},3-\frac{1}{n}\right)$ **Ans:** open

n) $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)$ **Ans:** closed

o) $\bigcup_{n=2}^{\infty}\left[\frac{1}{n},1-\frac{1}{n}\right]$ **Ans:** open

p) $\bigcap_{k=1}^{\infty}\left[0,\frac{1}{k}\right]$ **Ans:** closed

2. **(Interiors, Boundaries, and Exteriors)** Fill in the blanks in the following table.

	A	$\text{Int}(A)$	$\text{Bdy}(A)$	$\text{Ext}(A)$
a)	$\mathbb{Z}$			
b)	$\{\sin n : n \in \mathbb{N}\}$			
c)	$(0,\infty)$			
d)	$[0,1] \cup \{2\}$			
e)	$\{1,5,6\}$			
f)	$\{\sin x : 0 \le x \le \pi\}$			
g)	$(-1,0) \cup (0,1)$			
h)	$\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$			

Ans:

	A	$\text{Int}(A)$	$\text{Bdy}(A)$	$\text{Ext}(A)$
a)	$\mathbb{Z}$			
b)	$\{\sin x : 0 < x < 2\pi\}$	$(-1,1)$	$\{-1,1\}$	$\mathbb{R} - [-1,1]$
c)	$(0,\infty)$	$(0,\infty)$	$\{0\}$	$(-\infty,0)$
d)	$(0,1) \cup \{2\}$	$(0,1)$	$\{0,1,2\}$	$(-\infty,0) \cup (1,2) \cup (2,\infty)$
e)	$\{1,2\}$	$\varnothing$	$\{1,2\}$	$\mathbb{R} - \{1,2\}$
f)	$\{\sin x : 0 \le x \le \pi\}$	$(0,1)$	$\{0,1\}$	$\mathbb{R} - [0,1]$
g)	$(-1,0) \cup (0,1)$	$(-1,0) \cup (0,1)$	$\{-1,0,1\}$	$\mathbb{R} - [-1,1]$
h)	$\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$	$\varnothing$	$\{0\}$	$\mathbb{R} - \{0\}$

3. **(True or False)** Answer true or false about the following sets of real numbers.

a) A non empty set can be both open and closed.
Ans: True, $\mathbb{R}$ is both open and closed and it is the only non empty set that has this property**.**

b) A point can lie both in the interior and on the boundary of the set.
Ans: False, the interior, boundary, and exterior of a set are disjoint.

c) Finite sets are always closed.
Ans: True

d) Infinite sets are always open.
Ans: False, $[0,1]$ is a counterexample.

e) The boundary of a set is always finite.
Ans: False, the boundary of $\mathbb{Q}$ is $\mathbb{R}$

4. **(Mystery Sets)** Find two sets, which are not equal, but have the same interior, boundary, and exterior.
Ans: $[0,1]$ and $(0,1)$

5. **(Finding Examples)** Find the following sets of real numbers.
a) A set with two boundary points in the set and one boundary point not in the set.
Ans: $[0,1)\cup\{2\}$

b) A set with four boundary points in the set and three boundary points not in the set.
Ans: $[0,1]\cup(2,3)\cup[4,5)\cup\{6\}$

c) A set with three boundary points, none of which lie in the set.
Ans: $\left\{\frac{1}{n} : n\in\mathbb{N}\right\}\cup(2,3)$

d) A set with three boundary points, all of which lie in the set
Ans: $[0,1]\cup\{2\}$

6. **(Finite Sets Closed)** Show that the finite set $A=\{1,2\}$ is closed by finding its complement and showing that the complement passes the test of being an open set.

Ans: The complement is the union of three disjoint open intervals $\overline{A}=(-\infty,1)\cup(1,2)\cup(2,\infty)$. It is clear than for any $a\in\overline{A}$ there exists a small enough δ-neighborhood containing a that lies within $\overline{A}$.

7. **(Limit Points)** Find (if any) the limit points of the following sets. State if the conditions of the Bolzano-Weierstrass theorem hold.

a) $\mathbb{N}$
Ans: $\text{Limits}(\mathbb{N})=\varnothing$ The set is not bounded.

b) $\mathbb{Q}$
Ans: $\text{Limits}(\mathbb{Q})=\mathbb{R}$ The set is not bounded.

c) $\mathbb{R}$

Ans: $\text{Limits}(\mathbb{R}) = \mathbb{R}$ The set is not bounded.

d) $(2,4)\cup(4,5)$

Ans: $\text{Limits}\big((2,4)\cup(4,5)\big) = [2,5]$ Satisfies Bolzano Weierstrass

e) $\{(-1)^n : n \in \mathbb{N}\}$

Ans: $\text{Limits}\big(\{(-1)^n : n \in \mathbb{N}\}\big) = \varnothing$ Not infinite.

f) $\varnothing$

Ans: $\text{Limits}(\varnothing) = \varnothing$ Not infinite

g) $\mathbb{Q}\cap(0,1)$

Ans: $\text{Limits}\big(\mathbb{Q}\cap(0,1)\big) = [0,1]$ Satisfies Bolzano Weiterstrass

h) $\left\{\dfrac{m}{2^n} : m,n \in \mathbb{N}\right\}$

Ans: $\text{Limits}\left(\left\{\dfrac{m}{2^n} : m,n \in \mathbb{N}\right\}\right) = \mathbb{R}$ Not bounded

i) $\left\{m + \dfrac{1}{n} : m,n \in \mathbb{N}\right\}$

Ans: $\text{Limits}\left(\left\{m + \dfrac{1}{n} : m,n \in \mathbb{N}\right\}\right) = \mathbb{N}$ Not bounded

8. **(Closed Sets)** A set is closed if and only if it contains its limit points. Find the limit points of the following sets and determine if the sets are closed.

a) $\mathbb{Z}$

Ans: $\text{Limits}(\mathbb{Z}) = \varnothing$ Hence $\mathbb{Z}$ is closed.

b) $\mathbb{Q}$

Ans: $\text{Limits}(\mathbb{Q}) = \mathbb{R}$, hence $\mathbb{Q}$ is not closed.

c) $\mathbb{R}$

Ans: $\text{Limits}(\mathbb{R}) = \mathbb{R}$, hence $\mathbb{R}$ is closed.

d) $(2,4)\cup(4,5)$

Ans: $\text{Limits}\left((2,4)\cup(4,5)\right)=[2,5]$, hence the set is not closed.

e) $\left\{(-1)^n : n\in\mathbb{N}\right\}$

Ans: $\text{Limits}\left(\left\{(-1)^n : n\in\mathbb{N}\right\}\right)=\varnothing$, hence the set is closed.

f) $\varnothing$

Ans: $\text{Limits}(\varnothing)=\varnothing$, hence the empty set is closed

g) $\mathbb{Q}\cap(0,1)$

Ans: $\text{Limits}\left(\mathbb{Q}\cap(0,1)\right)=[0,1]$, hence the set is not closed.

h) $\left\{\dfrac{m}{2^n} : m,n\in\mathbb{N}\right\}$

Ans: $\text{Limits}\left(\left\{\dfrac{m}{2^n} : m,n\in\mathbb{N}\right\}\right)=\{0\}$, hence the set is not closed.

i) $\left\{m+\dfrac{1}{n} : m,n\in\mathbb{N}\right\}$

Ans: $\text{Limits}\left(\left\{m+\dfrac{1}{n} : m,n\in\mathbb{N}\right\}\right)=\mathbb{N}$, hence the set is not closed.

9. **(Intersections of Closed Intervals)** The intersection of a finite number of closed intervals is one of three types of sets. What are they?

Ans: empty set, one closed interval, one point

10. **(Intersections of Open Intervals)** The intersection of a finite number of open intervals is one of two types of sets. What are they?

Ans: empty set, one open interval

11. **(Examples)** Give examples of the following.

a) A bounded set with no limit points.
b) An unbounded set with one limit point.
c) A set with two limit points.
d) An unbounded set with whose limit points have cardinality $\aleph_0$.
e) An unbounded with one limit point.
f) An open set with no limit points.

a) **Ans:** $\{1,2,3\}$

b) **Ans:** $\mathbb{N} \cup \left\{ \frac{1}{n} : n = 2,3,\ldots \right\}$ Limit point is 0

c) **Ans:** $\left\{ (-1)^n \left(\frac{n-1}{n} \right) : n \in \mathbb{N} \right\}$ Limit points are ± 1.

d) **Ans:** $\left\{ m + \frac{1}{n} : m,n \cup \mathbb{N} \right\}$ Limit points are the natural numbers

e) **Ans:** $\left\{ \left(\frac{n-1}{n} \right) : n \in \mathbb{N} \right\} \cup \mathbb{N}$ Limit point is 1.

f) **Ans:** $\varnothing$

12. **(Sets and Limits)** Find examples of a set A of real numbers with the following properties:

a) A set that is equal to its limit points.
b) A set that is a subset of its limit points.
c) A set that contains all its limit points.
d) A set that is not a subset of its limit points and its limit points are not a subset of the set.

Ans:

a) $A = [0,1]$, $\text{limits}(A) = [0,1] \Rightarrow A = \text{limits}(A)$
b) $A = (0,1), \text{limits}(A) = [0,1] \Rightarrow A \subseteq \text{limits}(A)$
c) $A = \{1,2,3\}$, $\text{limits}(A) = \varnothing \Rightarrow \text{limits}(A) \subseteq A$
d) $A = (0,1) \cup \{2\}$, $\text{limits}(A) = [0,1] \Rightarrow A \not\subset \text{limits}(A) \wedge \text{limits}(A) \not\subset A$

13. **(Intersections and Unions of Closed Sets)** Show that the intersection of any family of closed sets is closed and that the union of a finite number of closed sets is closed. Hint: Use the properties of open sets and DeMorgan's laws.

Ans: Since any union

$$\bigcup_{\alpha \in \Delta} O_\alpha$$

of open sets O_α is open, its complement

$$\overline{\bigcup_{\alpha\in\Delta} O_\alpha} = \bigcap_{\alpha\in\Delta} \overline{O_\alpha}$$

closed, which shows that an arbitrary intersection of closed sets is closed.

Also, since a finite intersection

$$\bigcap_{k=1}^{n} O_k$$

of open sets is open, its complement

$$\overline{\bigcap_{k=1}^{n} O_k} = \bigcup_{k=1}^{n} \overline{O_k}$$

is closed, which shows a finite union of closed sets is closed.

14. **(Cantor Set)** Let $I = [0,1]$. Remove the open middle third

$$\left(\frac{1}{3}, \frac{2}{3}\right)$$

and call A_1 the set that remains; that is

$$A_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Now remove the open third intervals from each of these two parts of A_1, and call the remaining part A_2. Thus

$$A_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

Continuing in this manner, remove the open middle third of each segment in A_k and call the remaining set A_{k+1}. Note that we will get

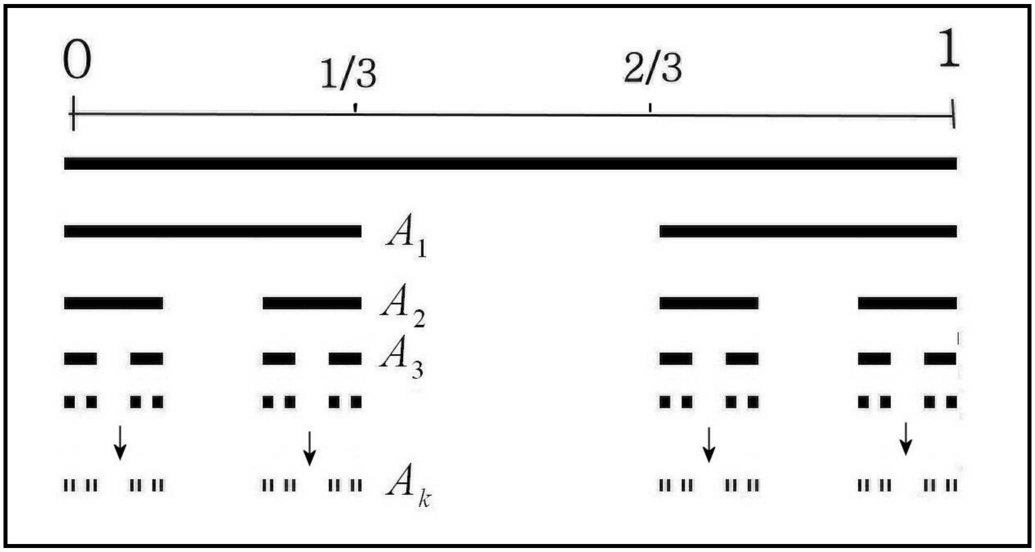
$$A_1 \supset A_2 \supset A_3 \supset \cdots A_k \supset \cdots$$

Continue this process indefinitely, always removing the middle third of existing segments. See Figure 1. The end set of this infinite process is called the Cantor set, and is defined as

$$C = \bigcap_{k=1}^{\infty} A_k .$$

a) Are there be any points left in the Cantor set?

b) Show the Cantor set is closed[1].



Cantor Set

Ans: complement of the Cantor set is a collection of disjoint intervals, each of which has finite length. Hence, any point in this set is contained in a neighborhood that is in the set. It has the interesting properties that it is uncountable but Lebesgue measure 0.

15. **(Intersection of Open Sets)** Find an example of a family of open sets whose intersection is not open.

Ans: $\bigcap_{n=1}^{\infty}\left(0,1+\frac{1}{n}\right)=(0,1]$

16. **(Union of Closed Sets)** Find an example of a family of closed sets whose union is not closed.

Ans: $\bigcup_{n=1}^{\infty}\left[\frac{1}{n},\, 2-\frac{1}{n}\right]=(0,2)$

17. **(Topologies on $\{a,b,c\}$)** A topology on a set X is a family of subsets of X that is closed under unions and finite intersections. Show whether the families $T_1,\ T_2 \in P(X)$ are or are not topologies on $X=\{a,b,c\}$.

a) $T_1=\{\varnothing,\{a\},\{a,b\},\{a,b,c\}\}$

[1] The Cantor set has a variety of interesting mathematical properties; has no interior, every point is an limit point, is uncountable but at the same time has total "length" (measure) zero..

Ans: $T_1 = \{\varnothing, \{a\}, \{a,b\}, \{a,b,c\}\}$ is a topology on $X = \{a,b,c\}$. The reader can verify that the union and intersection of any two sets in T_1 is again in T_1.

b) $T_2 = \{\varnothing, \{a\}, \{c\}, \{a,b\}, \{a,b,c\}\}$

Ans: $T_2 = \{\varnothing, \{a\}, \{c\}, \{a,b\}, \{a,b,c\}\}$ is not a topology on $X = \{a,b,c\}$ since $\{a\} \cup \{c\} = \{a,c\} \notin T_2$.

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