Section 6.1 Symmetries and Algebraic Systems

1. Determine the number of line and rotational symmetries of the following letters.

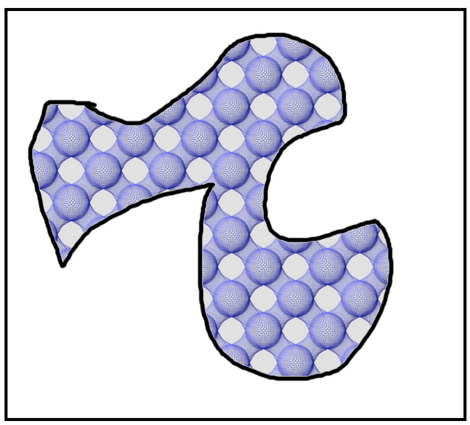
a)	A	l)	L	w)	W
b)	B	m)	M	x)	X
c)	C	n)	N	y)	Y
d)	D	o)	O	z)	Z
e)	E	p)	P		
f)	F	q	Q		
g)	G	r)	R		
h)	H	s)	S		
i)	I	t)	T		
j)	J	u)	U		
k)	K	v)	V		

Ans: We leave this fun exercise to the reader.

2. Draw a figure that has the following symmetries.

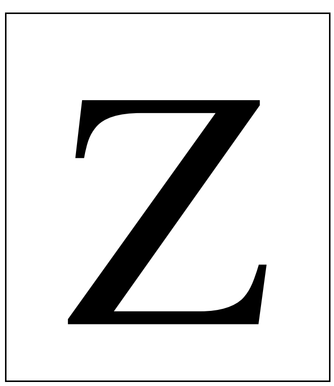
a) no rotational and no line symmetries

Ans:

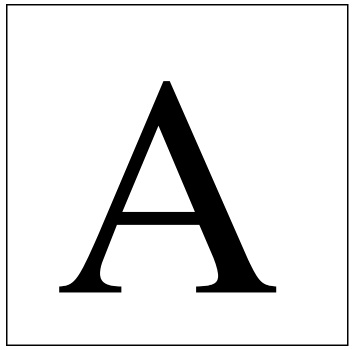


b) 1 rotational and no line symmetries

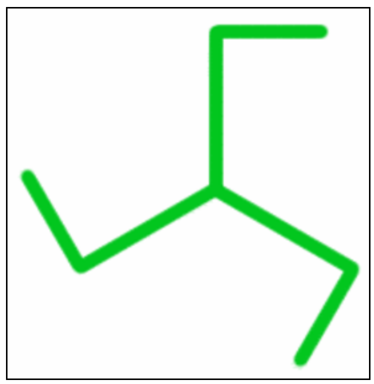
Ans:



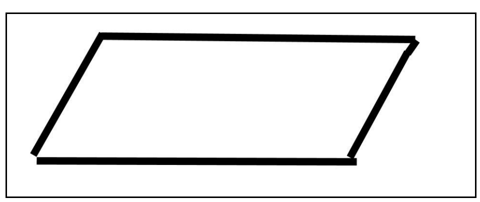
c) 1 rotational and 1 line symmetry
Ans:



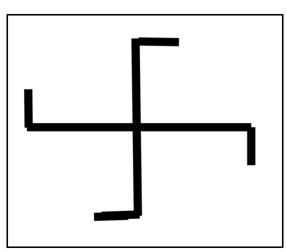
d) 3 rotational and no line symmetries
Ans:



e) One rotational and no line symmetries
Ans: Rotation by 180 degrees of a parallelogram



f) 4 rotational and no line symmetries
Ans:



3.**(Common Symmetries)**[1] The following logos have an equal number of line and rotation symmetries. These are the symmetries of a regular n gon. Symmetries of this type are called **dihedral symmetries**. Find the rotational and reflection symmetries of the following figures. Hint: Don't forget the identity mapping which is a rotation of zero degrees.

a)

Ans: One vertical line symmetry and a zero degree rotational symmetry.

b)

Ans: One horizontal and one vertical line symmetry, 2 rotational symmetries of 0 and 180 degrees.

c)

Ans: Three line symmetries, three rotational symmetries of 0 and 120 and 240 degrees.

d)

Ans: One vertical line symmetry, one horizontal, and two diagonal line symmetries, four rotational symmetries of 0, 90, 180, and 270 degrees.

e)

[1] We thank Annalisa Crannel of Franklin and Marshall College for providing these logos.

Ans: : Five line symmetries, five rotational symmetries of 0, 72, 144, 216, and 288 degrees.

4. **(Symmetries of Solutions of Differential Equations)** The solutions of the differential equation $dy/dx = y$ is the one-parameter family $y = ce^x$ where c is an arbitrary constant. Show that the transformation $x' = x + h,\ y' = y$ where h is an arbitrary real number maps the family of solutions back into the family of solutions, and hence is a symmetry transformation of the differential equation.

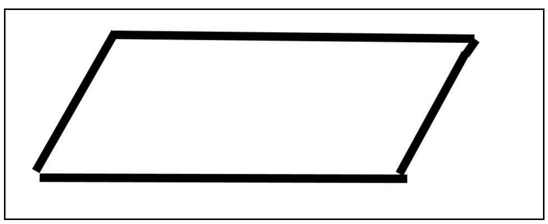
Ans: We begin with the family of curves $y = ce^x$, where c is an arbitrary real number and make the transformation $x = x' - h,\ y = y'$ getting

$$y' = ce^{(x'-h)} = ce^{x'}e^{-h} = c_1e^{x'}$$

where $c_1 = ce^{-h}$. Hence we have arrived at the new family in the coordinates x', y'.

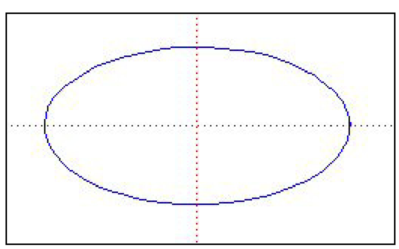
5. **(Symmetries of a Parallelogram)** Describe the symmetries of a parallelogram that is neither a rhombus nor a rectangle.

Ans: The only symmetry is a rotational symmetry of 180 degrees.



6. **(Symmetries of an Ellipse)** Describe the symmetries of an ellipse that is not a circle.

Ans: The ellipse has two line symmetries, horizontal and vertical, and two rotational symmetries of 0 and 180 degrees.



7. **(Representation of D_2 with Matrices)** Show that the matrices

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

satisfy the following multiplication table of the dihedral group D_2 where the group operation is defined as matrix multiplication. This means the group can be *represented* by matrices where the group operation is matrix multiplication.

$\circ$	e	A	B	C
e	e	A	B	C
A	A	e	C	B
B	B	C	e	A
C	C	B	A	e

Dihedral Multiplication Table

Ans: We leave it to the reader to carry out these matrix multiplications. For example

$$AB = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = C \ .$$

8. **(Representation of D_3 with Matrices)** Show that the following matrices satisfy the following multiplication table of the dihedral group D_3 of symmetries of a square, where the group operation is defined as matrix multiplication. This means the group can be *represented* by matrices where the group operation is matrix multiplication.

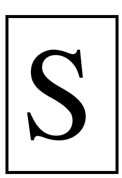
$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ R_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, R_{180} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ R_{270} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ F_{ne} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ F_{nw} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

Ans: We leave this routine computation for the reader.

9. **(Symmetry Groups)** Find the symmetries of the following figures and make a multiplication table for the symmetries.

a)



Ans:

	e	R_{180}
e	e	R_{180}
R_{180}	R_{180}	e

b)

T

Ans:

	e	V
e	e	V
V	V	e

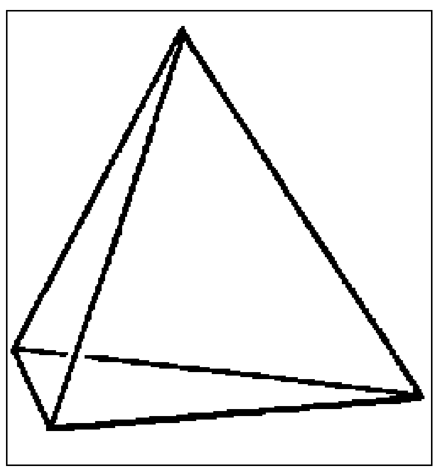
c)

J

Ans:

	e
e	e

10. **(Symmetries of a Tetrahedron)** Find the axes of symmetry and symmetries of a tetrahedron illustrated in the following figure.



Ans: There are a total of 7 axes of symmetry of the tetrahedron; 4 axes which pass through one vertex and the center of the opposite face, and 3 axes, each of which passes through the midpoints of opposite edges. The 4 axes which pass through a vertex and the opposite face have symmetry rotations of 120 and 240 degrees for a total of 8 symmetries, the 3 axes through midpoints of opposite edges have one rotation symmetry of 180 degrees for a total of 3 symmetries. Including the identity symmetry this gives a total of 12 rotation symmetries of a tetrahedron. Note that unlike the cube the tetrahedron does not have opposite faces and opposite vertices.

11. (**Cayley Table for** D_3) Note that the Cayley table for the symmetries of an equilateral triangle is bunched together into four distinct blocks; two blocks consists of rotations, two blocks of flips. Using this table tell which of the following are true and which are false.

a) A rotation followed by a rotation is a rotation; that is, $RR = R$
b) A rotation followed by a flip is a rotation; that is, $RF = R$
c) A rotation followed by a rotation is a flip; that is, $RR = F$
d) A rotation followed by a flip is a flip; that is, $RF = F$
e) A flip followed by a flip is a rotation; that is, $FF = R$
f) A flip followed by a rotation is a rotation; that is, $FR = R$
g) A flip followed by a flip is a flip; that is, $FF = F$
h) A flip followed by a rotation is a flip; that is, $FR = F$

Ans:

a) A rotation followed by a rotation is a rotation; that is, $RR = R$ **Ans:** T
b) A rotation followed by a flip is a rotation; that is, $RF = R$ **Ans:** F
c) A rotation followed by a rotation is a flip; that is, $RR = F$ **Ans:** F
d) A rotation followed by a flip is a flip; that is, $RF = F$ **Ans:** T
e) A flip followed by a flip is a rotation; that is, $FF = R$ **Ans:** T
f) A flip followed by a rotation is a rotation; that is, $FR = R$ **Ans:** F
g) A flip followed by a flip is a flip; that is, $FF = F$ **Ans:** F
h) A flip followed by a rotation is a flip; that is, $FR = F$ **Ans:** T

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