Section 6.3 Groups of Permutations: The Symmetric Group

1. Given the permutations

$$P=\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix},\quad Q=\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}$$

find:

a) PQ

Ans: $PQ=\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}$

b) $P^{-1}=\begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}$

Ans:

c) $QP^{-1}=\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$

Ans:

d) $P^2=\begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}$

Ans:

e) $(PQ)^{-1}$

Ans: $(PQ)^{-1}=\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}^{-1}=\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}$

2. **(Permutation Identity**) For permutations

$$P=\begin{pmatrix}a&b&c&d\\A&B&C&D\end{pmatrix},\quad Q=\begin{pmatrix}e&f&g&h\\E&F&G&H\end{pmatrix}$$

prove or disprove $(PQ)^{-1}=Q^{-1}P^{-1}$.

Ans: True by simple verification.

3. **(Cycle Notation)** Find the permutation

$$P=\begin{pmatrix}1&2&3&4&5\\?&?&?&?&?\end{pmatrix}$$

represented by the following cyclic products

a) (13)(24)

Ans: $P=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 4 & 1 & 2 & 5\end{pmatrix}$

b) (123)(45)

Ans: $P=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 3 & 1 & 5 & 4\end{pmatrix}$

c) (1432)

Ans: $P=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 1 & 2 & 3 & 5\end{pmatrix}$

d) (1)(2)(35)(4)

Ans: $P=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 2 & 5 & 4 & 3\end{pmatrix}$

e) (135)(42)

Ans: $P=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 4 & 5 & 2 & 1\end{pmatrix}$

4. **(Composition of Permutations)** For the following permutations

$$P=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 2 & 4 & 5 & 1\end{pmatrix},\quad Q=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 4 & 1 & 3 & 5\end{pmatrix},\quad R=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 5 & 3 & 1 & 4\end{pmatrix}$$

a) Show that $PQ \neq QP$

b) Verify $(PQ)R = P(QR)$

c) Verify $(PQ)^{-1} = Q^{-1}P^{-1}$

Ans: Direct computation

5. **(Cycles as the Product of 2-cycles)** A two-cycle is an exchange of two elements of a set, such as the permutation (23) of interchanging 2 and 3, leaving the other elements of the set unchanged. Every permutation of a finite set can be written (not uniquely) as the product of 2-cycles. Write the permutation (12345) as the product or composition of 2-cycles. Take five different objects and put them in a row and verify that your answer is correct by shuffling them in both ways.

Ans: $(12345) = (12)(13)(14)(15)$

6. **(Symmetric Group S_2)**

Given the set $A = \{1, 2\}$.

a) Construct the Cayley table for the group of permutations on A.
b) What is the order of this group?
c) Is the group Abelian?
d) What is the inverse of each element of the group?

Ans: a) The permutations are

$$e = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \; a = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

and the Cayley table for the group of permutations is

$\otimes$	e	a
e	e	a
a	a	e

b) The order is 2.
c) The group is Abelian.
d) The inverse of the identity is itself and the inverse of a is itself.

7. **(Transpositions)** Verify the products

$a)$ $(1234\cdots n) = (12)(13)(14)\cdots(1n)$
$b)$ $(214) = (21)(24) = (24)(12)$
$c)$ $(4321) = (43)(42)(41)$
$d)$ $(15324) = (15)(13)(12)(14)$

Ans: Direct computation.

7. **(Do Transpositions Commutate?)** Do transpositions commute in general? For example, for a set $\{1,2,3\}$ is it true that $(12)(13) = (13)(12)$?

Ans: No, we have

$$(12)(13) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \; (13)(12) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

8. **(Decomposition Into Transition is Not Unique)** Show that the decomposition of the permutation (12345) can be written as any of the three forms:

$$
\begin{aligned}
(12345) &= (12)(13)(14)(15) \\
&= (15)(25)(35)(45) \\
&= (23)(24)(25)(21)
\end{aligned}
$$

Ans: Simply carry out the computation in each of the four cases. It would be instructive if you line up five distinct objects and carry out each of the four shufflings.

Cartesian (or Direct) Product of Groups

It is possible to piece together smaller groups to form larger groups. If H and G are groups, their Cartesian (or direct) product[1] is defined to be the set

$$H \times G = \{(h, g) : h \in H, g \in G\}$$

where the group operation $*$ between members of $H \times G$ is

$$(h, g) * (h', g') = (hh', gg') \ .$$

The following problems illustrate some Cartesian products of groups.

9. **(Cartesian Product** $\mathbb{Z}_2 \times \mathbb{Z}_2$) Consider the cyclic groups $\mathbb{Z}_2 = \{0,1\}$ where the group operation is addition mod 2. Find the Cartesian product $\mathbb{Z}_2 \times \mathbb{Z}_2$ and construct its multiplication table. Show the table is the same as the multiplication table for the Klein four group of symmetries of a rectangle. In other words the Klein four group is **isomorphic** to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Ans: The members of the group are the Cartesian product $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$ where multiplication between the members is

$\oplus$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$

Note that this table is identical to the Cayley table for the Klein 4-group in Example 4 in Section 6.2 if we make the identification $e \leftrightarrow (0,0), a \leftrightarrow (0,1), b \leftrightarrow (1,0), c \leftrightarrow (1,1)$.

[1] The Cartesian (or **direct product**) product is often written $H \oplus G$.. The Cartesian product can be extended to the product on any number of groups, like $G_1 \times G_2 \times \cdots \times G_n$.

10. **(Cartesian Product $\mathbb{Z}_2 \times \mathbb{Z}_3$)** Find the elements of the Cartesian product $\mathbb{Z}_2 \times \mathbb{Z}_3$. What is the order of the group? What is the Cayley table for the group? Hint: Keep in mind that the product $(a,b)(c,d) = (e,f)$ where $e = (a+b) \bmod 2$, $f = (b+d) \bmod 3$.

Ans: The group $\mathbb{Z}_2$ has 2 members and $\mathbb{Z}_3$ has 3 members so the Cartesian product has 6 members, which are $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$. The Cayley table is

$\oplus$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,0)$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,1)$	$(0,1)$	$(0,2)$	$(0,0)$	$(1,1)$	$(1,2)$	$(1,0)$
$(0,2)$	$(0,2)$	$(0,0)$	$(0,1)$	$(1,2)$	$(1,0)$	$(1,1)$
$(1,0)$	$(1,0)$	$(1,1)$	$(1,2)$	$(0,0)$	$(0,1)$	$(0,2)$
$(1,1)$	$(1,1)$	$(1,2)$	$(1,0)$	$(0,1)$	$(0,2)$	$(0,0)$
$(1,2)$	$(1,2)$	$(1,0)$	$(1,1)$	$(0,2)$	$(0,0)$	$(0,1)$

12. **(Even and Odd Permutations and the 15-Puzzle)**
There are many ways to write a permutation as a composition of simple transpositions. For example, the following permutation of five elements can be written either as

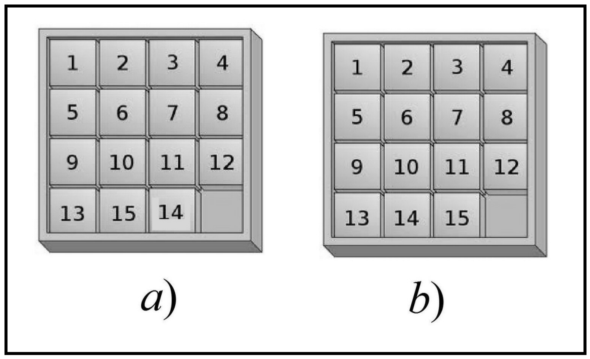
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} = (45)(35)(24)(12)(23)$$

or

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} = (24)(13)(15)$$

Although the above permutation can be written as the different compositions of transpositions, each of them consists of an odd number 1,3,5,… of transpositions . Permutation of this type is called **odd permutations**. On the other hand, an **even permutation** is one whose decomposition consists of an even number of transpositions. It can be shown that all permutations (of finite sets) are either even or odd. You can check to see that of the six permutations of the set $\{1,2,3\}$, three are even and three are odd.

The 15-puzzle consists of 15 numbered squares on a 4×4 grid with one of the squares missing. The idea is to start with the arrangement in the following figure a) and slide the squares around in any way you desire so that the 14-square and 15-square are interchanged with the blank square returned to the lower right as demonstrated in figure b). Show it is impossible to go from the arrangement in figure a) to the arrangement in figure b) by answering the following questions.



15-puzzle

a) Calling the blank square by square 16, convince yourself that every movement of a square is a transposition of the numbers $\{1,2,3,\ldots,15,16\}$.

b) Carry out the sequence of transpositions $(12\ 16)(11\ 16)(15\ 16)(12\ 16)$ to get a feel for the movement of the empty square.

c) Realize that to interchange squares 14 and 15 we must carry out the transposition $(14\ \ 15)$ on the set $\{1,2,3,\ldots,15,16\}$.

d) Realize that to interchange squares 14 and 15 we must carry out a series of n transpositions

$$(14\ 15) = (a_1\ 16)(a_2\ 16)\cdots(a_{n-1}\ 16)(a_n\ 16)$$

where $1 \le a_i \le 15$.

e) Realize that since the empty square 16 returns to its original position at the lower-right, it must move up and down and equal number of times, and left and right an equal number of times, thus making the total number n of transpositions and even number.

f) Observe that the equation

$$(14\ 15) = (a_1\ 16)(a_2\ 16)\cdots(a_{n-1}\ 16)(a_n\ 16)$$

cannot hold since there is 1 transposition on the left (odd number) and an even number of transpositions on the right hand side. Hence, it is impossible to move from the arrangement in Figure 9a) to the arrangement in Figure 9b).

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