Section 6.5 Rings and Fields

1. **(True or False)**

a) A ring can be finite or infinite.
Ans: true

b) In a ring $\{R, +, \times\}$ the set R with multiplication $\times$ forms a subgroup.
Ans: false

c) In a ring $\{R, +, \times\}$ the set R with addition $+$ is a group but doesn't have to be commutative.
Ans: false, it is an abelian group

d) The ring $\mathbb{Z}_{11}$ is also a field.
Ans: true

e) The ring $\mathbb{Z}_8$ is also a field
Ans: false

f) There are fields where $a \times b = 0$ but neither a or b are zero.
Ans: false, this maybe be true in some rings but not fields.

2. **(Multiplicative Identity)** For each of the following rings, tell if the ring is commutative and if there exists a multiplicative identity. If a multiplicative identity exists, what is it?

a) The ring of integers $\mathbb{Z}$ with usual addition and multiplication
Ans: The ring is commutative with multiplicative identity 1.

b) The ring of even integers $2\mathbb{Z}$ with usual addition and multiplication.
Ans: The ring is commutative but it does not have a multiplicative identity.

c) The ring $C(\mathbb{R})$ of real-valued continuous functions defined on $\mathbb{R}$.
Ans: The ring is commutative with multiplicative identity $f(x) = 1$.

d) The ring of the sets $\mathbb{Z}\left[\sqrt{2}\right] = \left\{m + n\sqrt{2} : m, n \in \mathbb{Z}\right\}$ with usual addition and multiplication.
Ans: The ring is commutative with multiplicative identity 1.

e) The ring $\mathbb{Z}[x]$ of all polynomials in x with integer coefficients with ordinary addition and multiplication.
Ans: Yes the ring is commutative with multiplicative identify $f(x) = 1$.

f) The ring $\mathbb{Q}$ of rational numbers with ordinary addition and multiplication.

Ans: Yes, it is commutative and the multiplicative identity is 1. In fact $\mathbb{Q}$ is a field.

g) The ring consisting of the set $\mathbb{Z}_3 = \{0,1,2\}$ where addition and multiplication are defined modulo 3.

Ans The ring is commutative and has multiplicative identity 1. In fact $\mathbb{Z}_3$ is a field.

3. **(Rings Lacking Properties)** Find rings which lack the given property.

a) Ring without multiplicative identity**.** **Ans:** even integers $2\mathbb{Z}$.

b) Ring without multiplicative commutativity. **Ans:** matrices.

c) Ring without multiplicative inverse. **Ans:** integers.

4. **Ring of Matrices)** Show that the set of all 2×2 matrices

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z} \right\}$$

is a ring under matrix addition and matrix multiplication is a ring with a multiplicative element, but it is not commutative and it is not a field.

Ans: Not a field since not all matrices (i.e. singular matrices) have multiplicative inverses.

5. (**Rings which are not Fields)** Why are the following rings not fields ?

a) The ring of polynomials with real coefficients with the usual addition and multiplication.

b) The ring of $n \times n$ matrices with the usual matrix addition and multiplication.

c) The set $\mathbb{Z}_n = \{0,1,2,...,n-1\}$, where addition and multiplication are performed modulo n, where n s a composite number.

Ans:

a) The ring of polynomials with real coefficients with the usual addition and multiplication is a ring but not a field since polynomials do not in general have multiplicative inverses. For example $p(x) = x^2 + 2x + 1$ has no multiplicative inverse (i.e. no polynomial $q(x)$ so that $p(x)q(x) = 1$ .for all x.

b) The ring of $n \times n$ matrices with the usual matrix addition and multiplication is a ring, but not a field since matrices with zero determinant do not have multiplicative inverses.

c) The set $\mathbb{Z}_n = \{0,1,2,...,n-1\}$, where addition and multiplication are performed modulo n, where n s a composite number (not a prime) is a ring but not a field. Example 8 illustrates this idea for composite $n=4$.

6. (**Mod 3 Field**) The addition and multiplication tables for $\mathbb{Z}_3$ is shown below. What are the additive and multiplicative inverses (if they exist) for every member of the field.

+	0	1	2		×	0	1	2
0	0	1	2		0	0	0	0
1	1	2	0		1	0	1	2
2	2	0	1		2	0	2	1

Ans: The additive and multiplicative inverses of the members of the field are displayed in the table

a	$-a$	a^{-1}
0	0	-
1	2	1
2	1	2

For instance 0 has no multiplicative inverse since only the non-zero elements of the field are considered in the multiplication table.

7. (**Arithmetic in** $\mathbb{Z}_3$) In the field $\text{GF}(3) = \mathbb{Z}_3$ compute the following

a) $1+2$ **Ans:** 0

b) $1-2$ **Ans:** If $x = 1-2$, we seek an x that satisfies $x+2=0$. From the addition table for $\mathbb{Z}_3$ we see $x = 1-2 = 2$.

c) 2×2 **Ans:** 1

d) $1/2$ **Ans:** $1/2 = 1\times 2^{-1} = 1\times 2 = 2$

8. (**Modular Arithmetic**) Find an integer x such that

a) $2x = 1(\text{mod}\,3),\ x\in\mathbb{Z}_3$ **Ans:** $x = 1\times 2^{-1} = 1\times 2 = 2$

b) $3x = 2(\text{mod}\,5),\ x\in\mathbb{Z}_5$ **Ans:** $x = 2\times 3^{-1} = 2\times 2 = 4$

c) $4x = 3(\text{mod}\,7),\ x\in\mathbb{Z}_7$ **Ans:** $x = 3\times 4^{-1} = 3\times 2 = 6$

9**. Galois Field** $\text{GF}(2^2)$ Verify the addition and multiplication tables for $\text{GF}(2^2)$ in Table 4.

10. (**Multiplicative Inverse**) The integers $\mathbb{Z}$ under ordinary addition and multiplication form a commutative ring with unity 1. Do any members of this ring have multiplicative inverses ? If so, what are they ?

Ans: 1 and -1 have multiplicative inverses, namely themselves since $(1)(1)=1$ and $(-1)(-1)=1$.

11. **(Type of Ring)** The set $\{0,a,b,c\}$ with addition and multiplication defined by the following Cayley tables forms a ring. Is this group commutative and does it have a multiplicative identity?

⊕	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

⊗	0	a	a	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Ans: No to both. Commutative refers to the multiplication table which is clearly not commutative.

Zero Divisors and Integral Domains In some rings things don't obey the arithmetic you learned in grade school. For example, in the ring $\mathbb{Z}_4 = \{0,1,2,3\}$ modulo arithmetic we found $2 \times 2 = 0$. Here we say that 2 is a zero divisor for this ring. In general, an element $a \in R$ in a ring is a **zero divisor** if there is a nonzero element $b \in R$ in the ring such that $ab = 0$. Matrix rings also have zero divisors.

12. **(Zero Divisors)** Are there zero divisors in the ring of 3×3 matrices with integer entries using the usual operations of addition and multiplication?

Ans: Yes, the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

since

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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