A Taste of Pure Mathematics

by

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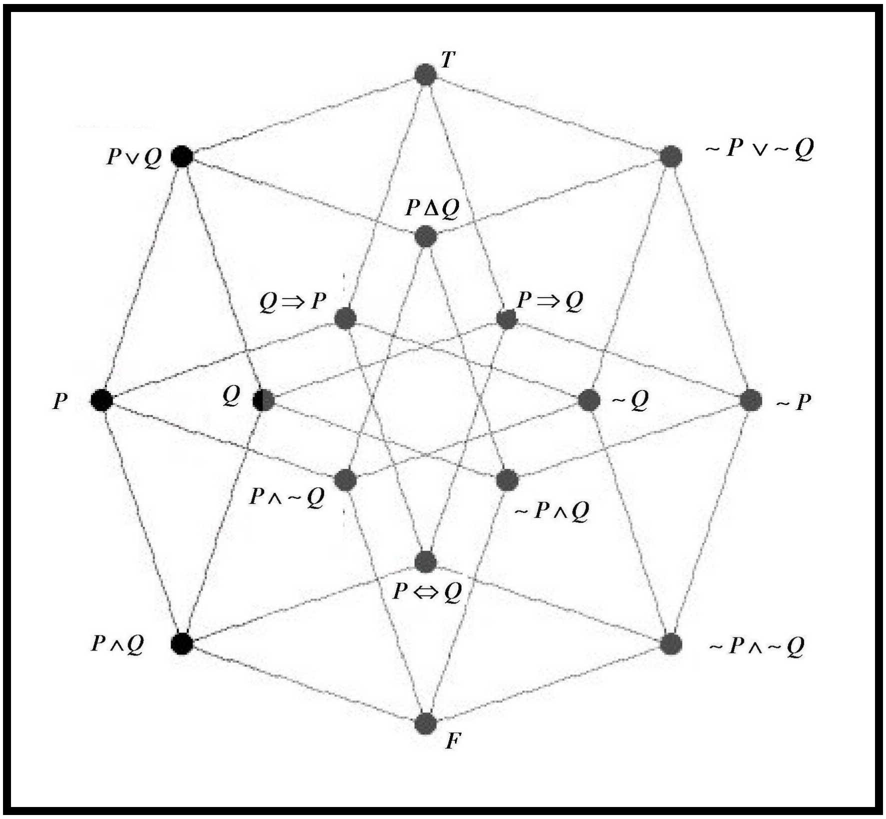

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Preface

This book is the culmination of a set of notes that has been growing and fermenting in a bottom drawer of my desk for over thirty years. The goal was that after the right amount of pedantic aging, they would be ready to bottle and share with the world. It is my hope that the finished product harbors the perfect blend of intuition and rigorous preciseness, theory versus pragmatism, hard-nosed mathematics and an enjoyable learning experience.

I must confess that when the entrails of this book were taking form the intended audience was the 25 or so students in my *Introduction to Higher Math* class, but now that they have been released into the wild, my hope is they will find a wider audience.

Possible Beneficial Audiences

• **College students** who have taken courses in calculus, differential equations and perhaps linear algebra, may not be prepared for the more advanced courses of real analysis, abstract algebra, and number theory that lie ahead. (If you can't negate a logical sentence, you're not ready for prime time.) Although the basic calculus et al sequence is important for developing a rounded background in mathematics as well as developing problem-solving skills, they are not intended to prepare the student for advanced mathematics. Few students, after going through the basic sequence, are even familiar with the basic language of mathematics, such as the sentential and predicate logic. If a student is to develop skills for reading, analyzing, and appreciating mathematical arguments, knowledge of the basic language of mathematics is a must.

• **Mathematics teachers and math education students** will hopefully find this book to be a valuable aid for teaching an inspiring and exciting introduction to mathematics that goes beyond the basics. The large and varied collection of historical notes and varied problems at the end of each section should be worth the "price of admission" alone.

• **High school students** with good backgrounds and a strong interest in mathematics will hopefully fall in love with the book, all the way from the "Important Note" inserts to the unusual problem set.

• **Scientists, engineers, and out-of-practice others** in the professional workforce, who are discovering that the mathematics they learned long ago in college is not adequate for their current needs, are taken through a crash course in beginning pure mathematics. This book might even show these self-study individuals just how uncomplicated and enjoyable mathematics can be, possibly at variance from their college experience. They might even develop newly found mathematical confidence, filling in gaps in their mathematical background, like what constitutes a mathematical proof, the language of mathematics, infinity, relations, topology, abstract algebra, and the like.

Wow Factors of the Book

Although this book is intended to raise the mathematical thought process of the reader, it is not intended for the reader to come away thinking that mathematics consists simply in reading and proving theorems. The goal is to blend a nuanced amalgam of inductive and deductive reasoning. Utmost in my mind was to avoid the dogmatism of page-after-page of theorems and proofs, to prompt the reader in thinking about the ideas presented.

The book is loaded with informative sidebars, historical notes, and tons of graphics, which hopefully provides an enjoyable atmosphere for a constructive learning process.

Over the years, I have managed to track down, dig up, sniff out, and even make up a few fascinating problems myself. Each section of the book is packed with loads of problems for readers of all interests and abilities. Each section is also filled with a broad range of examples intended to add clarity and insight to abstract concepts. I take my time to explain the thinking and intuition behind many concepts.

Chapter by Chapter (the nitty gritty)

Chapter 1: Logic and Proofs

This book is not *Principia Mathematica* by Alfred North Whitehead and Bertrand Russell, who famously prove 1+1=2 after 378 arduous pages in their seminal 1910 work on the foundations of mathematics. That said, there are many mathematical proofs in this book, and each and every one of them is intended to act as a learning experience. The first and foremost, of course, being that before anything can be proven true or false, it must be stated in a precise mathematical language, predicate logic. Chapter 1 introduces the reader to sentential and predicate calculus and mathematical induction. After two introductory sections of sentential logic and the connectives -- and, or, not, if then, if and only if -- we move up the logical ladder to predicate logic and the universal and existential quantifiers and variables. Sections 1.4 and 1.5 are spent proving theorems in a variety of ways, including direct proofs, proofs by contrapositive, and proofs by contradiction. The chapter ends with mathematical induction.

Chapter 2: Sets and Counting

Sets are basic to mathematics, so it is natural that after a brief introduction to the language of mathematics, we follow with an introduction to sets. Section 2.1 gives a barebones introduction to sets, including the union, intersection, and complements of a set. Section 2.2 introduces the reader to the idea of families of sets and operations on families of sets, tools of the trade for more advanced mathematical subjects like analysis and topology. Section 2.3 is an introduction to counting, including permutations, combinations, and the pigeonhole principle. Sections 2.4, 2.5, and 2.6 introduce the reader to the basics of Cantor's discoveries on the cardinality of infinite sets. Brief discussions are included on the continuum hypothesis and the axiom of choice, as well as an intuitive discussion of the Zermelo-Frankel axioms of set theory.

Chapter 3: Relations

As the English logician Bertrand Russell once said, mathematics is about relations. This chapter introduces order relations, e.g. partial, strict, and total order, followed by equivalence relations and the function relations. Section 3.5 introduces the idea of the image and inverse image of a set, concepts important in analysis and topology.

Chapter 4: The Real and Complex Number Systems

Sections 4.1 and 4.2 show how the real numbers can either be defined axiomatically, or constructed all the way from the natural numbers, to the integers, to the rational numbers, to the real numbers using equivalence relations introduced in Chapter 3, as well as the Dedekind cut. Section 4.3 gives a brief tutorial on the complex numbers, a subject often overlooked in undergraduate curricula.

Chapter 5: Topology

This chapter introduces a number of sides to topology, starting with Sections 5.1 and 5.2 on graph theory. There was considerable internal consternation of where, if at all, to include such material. At one time, an entire chapter on combinatorics was considered, but scrapped. It was finally decided to include a couple sections on graph theory in a chapter on topology. Section 5.3 introduces basic ideas, such as homeomorphisms, topological equivalence, Euler's equation, and a verification of Euler's equation for the Platonic solids. Finally, Section 5.5 introduces the basic ideas of point-set topology required for hard analysis, including open and closed sets, limit points, interior, exterior and boundary of a set, and so on.

Chapter 6: Algebra

Chapter 6 provides a brief introduction to symmetries and abstract groups, rings and fields. Section 6.1 finds the symmetries of various geometric objects and shows that a composition of symmetry functions gives rise to the algebraic structure of a group. Sections, 6.2, 6.3, 6.4 introduce symmetry and cyclic groups of permutations, along with the idea of a subgroup. Finally, Section 6.5 gives a brief summary of rings and finite fields.

Conclusion

May the ghosts of past students of this book be with you as you find your way to a mathematical utopia. In proper hands, the book can be used in a one-semester course, covering the entire book or portions of the book. For a more leisurely pace, an instructor can pick and choose depending on one's preferences. If rushed, one may pass over portions of the book, depending on one's preferences.

Enjoy.

... Stanley (Jerry) Farlow

Note to the Reader

And here let me insert a parenthesis here to insist on the importance of written exercises. Compositions in writing are perhaps not given sufficient prominence in certain examinations. In the École Polytechnique, I am told that insistence on compositions would close the door to very good pupils who know the subject, yet are incapable of applying it in the smallest degree. The word understand has several meanings. Such pupils understand only in the first sense of the word, and this is not sufficient to make either an engineer or a geometrician.

..Henri Poincaré

Keeping a Scholarly Journal

One cannot help but be impressed with the huge number of important English naturalists who lived during the nineteenth century. In addition to Darwin, there were Wallace, Eddington, Thompson, Haldane, Galton, and others. One characteristic that permeated their work was the keeping of detailed journals where every observation and impression was recorded. In addition to recording data, a journal provides a way to organize one's thoughts, explore relationships and formulate ideas. In fact, they stimulate learning through writing.

Journal keeping has declined in the 20th and 21st centuries, but readers of this book have the ability to recapture that important tool of learning-through-writing.'

Entries in your (leather-bound) journal can be entered daily or in conjunction with each section of the book. It is useful to date entries and give them short descriptive titles. While there are no specific rules on what to include in the journal or how to write them, you will eventually find your 'voice' on what works best. From the habit of rereading old entries each time you write new ones, see how your grasp of material grows. After years have passed, you will be impressed on the value you give past journal entries.