Section 1.1 Sentential Logic

Purpose of Section: To introduce **sentential (sen-TEN-shuhl) logic** and the fundamental idea of a **sentence** (or **proposition**) and show how simple sentences can be combined using the **logical connectives** "**and**", "**or**" and "**not**" to form **compound sentences**. We then analyze the meaning of these compound sentences by means of **truth tables** and then introduce the concepts of **logical equivalence**, **tautologies** and **contradictions**. We close by introducing the **conjunctive** and **disjunctive** forms of logical expressions.

Introduction:

So what is mathematics? The word itself is derived from the Greek word *mathēmatikē,* meaning "knowledge" or "learning." To both practitioners of mathematics as well as the general public, the definition of the Queen of the Sciences varies widely.

One of the greatest mathematician of Greek antiquity, Aristotle (384-322 BC) defined mathematics as:

Mathematics is the science of quantity.

Later, the Italian physicist Galileo (1546-1642), who was more interested in how it was applied, wrote:

"Mathematics is the language of the Universe and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it."

A more generic definition is given by the encyclopedia Britannica, which defines mathematics as

Mathematics is the science of numbers and shapes and the relations between them.

Then there is the beauty in mathematics as observed by the physicist Albert Einstein, who wrote

Pure mathematics is, in its way, the poetry of logical ideas.

At this point, the reader might take a moment to add his/her two cents to the huge pile of opinions about the *Queen of the Sciences*. It would be interesting to compare your own thoughts with those of fellow-travelers reading this book.

Some pure mathematicians prefer a more logical definition, constructed from logic and proofs, such as the one stated by Bertrand Russell in his book, The Principles of Mathematics, were he writes:

Mathematics is the collection of statements of the form, if P then Q.

In other words, an argument where if a certain statement P is assumed true, then applying accepted rules of logic and known mathematical facts, the statement Q is proven true.

Getting into Sentential Logic

Although mathematics uses symbolic notation, its arguments are formulated in natural languages, such as English, thus it becomes necessary to examine the truth value of different natural language sentences and understand how they can be combined to form more complex sentences. We begin our study with **sentential** (or **propositional**) **logic**), which is the most basic formal system of logic[1] and which uses **symbols** and **rules of inference** such as

if P is true and P implies Q, then Q is true[2]

Sentential logic is generally the first topic introduced in a basic course in logic, followed by more involved systems of logic, like predicate logic and modal logic.

> **Important Note:** The study of logic had its origins in many ancient cultures, but it was the writings of the Greek logician Aristotle (384-322 B.C.) who most influenced Western culture in a group of works known collectively as the *Oranon.*

English, as do all natural languages, contains various types of sentences, such as declarative, interrogative, exclamatory, and so on, which allow for the effective communication of thoughts and ideas. Some sentences are short and to the point, whereas others are long and rambling. Some sentences are either true or false, such as something one might hear on the weather channel that announces a storm approaching. Although one cannot be sure of the accuracy of such a statement, it is nevertheless true or false. Other sentences, like the interrogatory sentence, “Why doesn't Burger King sell hotdogs?” or the exclamatory sentence, “Don't go there!” express thoughts, but have no truth or

[1] There are other formal systems of logic other than propositional logic, such as **predicate** and **modal** logic. Because the rules of formal logic are precise, they can be programmed for a computer, and are capable of analyzing mathematical proofs. Formal logical systems are important in artificial intelligence, where computers are programmed in the language of formal logic to carry out logical reasoning.

[2] Greek logicians called this rule of inference *modus ponens*, meaning the *way that affirms*.

false value. The types of sentences we study in this book are declarative and are intended to convey information. In logic, the word "sentence" is used in a technical sense as described in the following definition.

Definition: A **sentence** (or **proposition**) is a statement which is either true or false. If the sentence is true, we denote its truth value[3] by the letter T, and by F if it is false In computer science they are sometimes denoted by 1 and 0, respectively..

Example 1 Are the following statements sentences or not sentences?

a) $\frac{17}{231} - \frac{4}{10}$ is a positive number

b) $\int_0^1 e^x dx = e$ is false.

c) $\sqrt{2}$ is a rational number.

d) Love is sharing your popcorn.

e) Come here!

f) N is an even integer.

g) Who first proved that π is a transcendental number?

h) Who was the greatest mathematician of the 20th century?

i) This sentence is false.

Solution

a)-d) are sentences: a) is false, b) is true, c) is false, d) the reader can decide.

e) The statement has no truth value so it is not a sentence.

f) Granted the statement *is* true or false, *but* its truth value depends on the value of an unknown number *N*, so it is not considered a sentence. In Section 1.3 we will introduce *quantifiers* which will turn this statement into a sentence.

g) It is an interesting question, but questions are not sentences. The person who first proved that π is transcendental was the German mathematician Ferdinand von Lindemann who proved it in 1882. The number π is also irrational, which was demonstrated by another German mathematician Johann Lambert in 1768.

h) This statement is not a sentence, and its answer is subject to debate. You can find candidates for the greatest mathematician by "googling" the phrase "famous mathematicians of the 20th century".

i) If the statement is true, then according to what the statement says, it is false. On the other hand, if one claims the statement is false, then the statement says it

[3] Sometimes truth and falsity are denoted by 1 and 0, respectively.

is true. In either case we reach a contradiction. Hence, we conclude the statement is neither true nor false, and hence not what we call a *sentence*. This paradox is one of the many forms of what is called Russell's paradox.

Liar Paradox: Consider the following sentence

This sentence is not true.

This sentence is paradoxical since if we say the sentence is true, the sentence itself says it's false, which yields a contradiction. On the other hand, if we say the sentence is false, the sentence says it's false, hence it must be true. In either case, one is led to a contradiction, so one cannot say the sentence is true or false.

Russell's Paradox: Russell's paradox is another example of a statement that refers to itself, called *self-referential* statements. Russell's **barber paradox**, considers a barber in a small town that *shaves all men in the town, but only those men, who do not shave themselves.* The prophetic question then arises, does the barber shave himself? If you say the barber shaves himself, then barber doesn't shave himself since he only shaves those who don't shave themselves. On the other hand, if you say the barber does *not* shave himself, then he shaves himself since he shaves those who don't shave themselves. This paradox was formulated by the English logician Bertrand Russell (1872-1970) in 1901, and played a major role in the modern development set theory.

Compound Sentences ("AND", "OR" and "NOT")

In arithmetic we combine and modify numbers with operations like $+, \times, -$ and so on. In logic, we do the same but with logical expressions. The sentences discussed thus far are examples of **simple** (or **atomic**) **sentences** since they are made up of a single thought or idea. It is possible to combine these simple sentences to form **compound sentences** using **logical connectives**[4].

Important Note: It has been said that love and the ability to reason are the two most important human traits. Readers interested in the first of these traits must go elsewhere for advice, but if the reader is interested in reasoning, this is the place to be.,

$\wedge$ means "and"
$\vee$ means "or"

[4] We can also combine compound sentences to form even more complex sentences.

Definition: Logical Connectives Given the sentences P and Q, we define:

Logical AND: The **conjunction** of P and Q , denoted $P \wedge Q$, is the sentence "*P* and *Q*" (logical AND) which is true when both P and Q are true, otherwise false.

Logical OR: The **disjunction** of P and Q , denoted $P \vee Q$, is the sentence "*P* or *Q*" (logical OR) which is true when *at least one* of P or Q are true, otherwise false

NOT operator: The **negation** (or **denial**) of P, denoted $\sim P$, is the sentence "not *P*" and $\sim P$ is true when P is false, and $\sim P$ is false when P is true.

Example 2: Logical Conjunction Let P and Q be the sentences

P : "Jack went up the hill"
Q : "Jill went up the hill"

The conjunction $P \wedge Q$ refers to the English sentence

"Jack and Jill went up the hill"

The truth of $P \wedge Q$ depends on the truth values of the two simple sentences P and Q. The conjunction $P \wedge Q$ is true if *both* Jack and Jill went up the hill, and false if *either* Jack or Jill did *not* go up the hill. We summarize this symbolically by means of a **truth table**, which examines the truth value of $P \wedge Q$ for the four possible truth values of P and Q. The two columns at the left list the possible truth values for the sentences P and Q.

P	Q	$P \wedge Q$	
T	T	**T**	$\wedge$ is true if both P and Q are true, otherwise false.
T	F	**F**	
F	T	**F**	
F	F	**F**	

Truth table for the logical AND
Table 1

Important Note: Because the rules of formal logical systems (like sentential logic, predicate logic, …) are precisely defined, it is possible to program them into a computer and get the computer to evaluate proofs quickly and automatically. This is important in artificial intelligence in computer science.

Example 3: Logical Disjunction Again let

P : "Jack went up the hill"
Q : "Jill went up the hill"

The disjunction $P \vee Q$ refers to the English sentence

"Jack or Jill went up the hill"

The disjunction $P \vee Q$ is true if *either* Jack *or* Jill (or both) went up the hill. If *neither* Jack *nor* Jill went up the hill the disjunction is false. This is summarized in the following truth table.

<table>
<tr><th>P</th><th>Q</th><th>$P \vee Q$</th><td rowspan="5">$\vee$ is false if both P and Q are false, otherwise true</td></tr>
<tr><td>T</td><td>T</td><td>T</td></tr>
<tr><td>T</td><td>F</td><td>T</td></tr>
<tr><td>F</td><td>T</td><td>T</td></tr>
<tr><td>F</td><td>F</td><td>F</td></tr>
</table>

Logical OR.
Table 2

Sometimes, in normal English discourse, the word "or" is used in an **exclusive** sense. For example, when someone says "*for dessert I will have pie or cake*" it is normally understood to mean the person will have exactly one of the two desserts, not both. In this case we would say "or" is an **exclusive OR**. In sentential logic, however, unless otherwise stated "or" means the **inclusive** OR as defined in Table 2.

Example 4: Negation of a Sentence If

P : "Jack went up the hill"

then $\sim P$ is the sentence "Jack did *not* go up the hill." In other words if P is true, then $\sim P$ is false, and if P is false then $\sim P$ is true . This is summarized by the following truth table.

P	$\sim P$
T	**F**
F	**T**

Truth table for negation
Table 3

Important Note: To check the accuracy of your truth tables, there are many online web pages that will carry out these computations.

Historical Note: There are theorems and then there are *theorems*. In the *Classification Theorem for Simple Groups* (known lovingly at the "*enormous*" theorem). The proof required the work of hundreds of mathematicians and consists of an aggregate of hundreds of papers. If the theorem were to be written out it is estimated it would take between 10,000 to 15,000 pages.

Compound Sentences

We can also combine sentences to form more sentences as the following examples show.

$P \wedge \sim Q$	Jack went up the hill but Jill did not.
$\sim (P \vee Q)$	Neither Jack nor Jill went up the hill.
$\sim (P \wedge Q)$	It isn't true that both Jack and Jill went up the hill.
$\sim P \vee Q$	Either Jack did not go up the hill or Jill did.

Forming new sentences
Table 4

The truth values of $\sim P \vee Q$ can be analyzed using the following truth table. The numbers (1) and (2) above the columns give the order in which the columns were filled in.

		(1)	(2)
P	Q	$\sim P$	$\sim P \vee Q$
T	T	F	**T**
T	F	F	**F**
F	T	T	**T**
F	F	T	**T**

Truth table for $\sim P \vee Q$
Table 5

Note that $\sim P \vee Q$ is false only when P is true and Q is false, otherwise it is true.

The compound sentence $(P \vee Q) \wedge \sim R$ contains three sentences *P, Q* and *R* and its truth value is determined by enumerating the $2^3 = 8$ possible truth values for *P, R*, and *R,* then examining the truth value of $(P \vee Q) \wedge \sim R$ in each case. The results are given in the truth table in Table 6.

			(1)	(2)	(3)
P	Q	R	$P \vee Q$	$\sim R$	$(P \vee Q) \wedge \sim R$
T	T	T	T	F	**F**
T	T	F	T	T	**T**
T	F	T	T	F	**F**
T	F	F	T	T	**T**
F	T	T	T	F	**F**
F	T	F	T	T	**T**
F	F	T	F	F	**F**
F	F	F	F	T	**F**

Truth table for $(P \vee Q) \wedge \sim R$

Table 6

Historical Note: In 1666 the philosopher and mathematician **Gottfried Wilhelm Leibniz** (1646-1716) laid out a plan in his *work De Arte Combinatorial* in which all reason could be reduced to mental calculations. He wrote, “The method should serve as a universal language whose symbols and special vocabulary can direct reasoning in such a way that errors, except for fact, will be mistakes in computation.” It is tragic that at the time, Leibniz’ ideas were accepted as fantasy and his ideas sank into oblivion. It was not until 1903, well after symbolic logic had been "rediscovered" by George Boole, Augustus DeMorgan and others, that Leibniz' pioneering research was introduced to the general audience.

An example of a sentence containing four component sentences is $(P \wedge Q) \vee (R \wedge \sim S)$, which depends on *P, Q ,R* and *S.* The truth value of this sentence is found by examining a truth table with $2^4 = 16$ rows listing all possible truth values of the four components. Table 7 shows how this truth table is computed.

				(1)	(2)	(3)	(4)
P	Q	R	S	$\sim S$	$P \wedge Q$	$R \wedge \sim S$	$(P \wedge Q) \vee (R \wedge \sim S)$
T	T	T	T	F	T	F	**T**
T	T	T	F	T	T	T	**T**
T	T	F	T	F	T	F	**T**
T	T	F	F	T	T	F	**T**
T	F	T	T	F	F	F	**F**
T	F	T	F	T	F	T	**T**
T	F	F	T	F	F	F	**F**
T	F	F	F	T	F	F	**F**
F	T	T	T	F	F	F	**F**
F	T	T	F	T	F	T	**T**
F	T	F	T	F	F	F	**F**
F	T	F	F	T	F	F	**F**
F	F	T	T	F	F	F	**F**
F	F	T	F	T	F	T	**T**
F	F	F	T	F	F	F	**F**
F	F	F	F	T	F	F	**F**

Truth table for $(P \wedge Q) \vee (R \wedge \sim S)$.

Table 7

Historical Note: After Leibniz the next major development in propositional logic was due to the work of English logicians George Boole (1815-1854) and Augustus DeMorgan (1806-1871). Boole was interested in developing a logical algebra, whereby the symbols x, y represented sets[5] or classes, where the empty set was denoted by 0, the universal set denoted by 1, intersection of sets by xy, and union of sets by $x + y$. Boole interpreted $x = 1$ to mean "x is true" and $x = 0$ as "x is false". In this system $xy = 1$ means x and y are both true, $x + y = 1$ means x or y is true, and so on. Boole's ideas sparked immediate interest among logicians and Boole's **Boolean algebras** are the basis for sentential logic and are used today in computer science and design of circuits.

[5] We are getting a little ahead of ourselves, but we will get to sets in Chapter 2.

Equivalence, Tautology, and Contradiction

Mathematical proofs require the ability to write a string of sentences whose truth values are the same. This brings us to the concept of logical equivalence.

> **Definition:** Two sentences P, Q simple or compound, are **logically equivalent**, denoted by $P \equiv Q$, if they have the same truth tables for all truth values of their component parts.

It is not always necessary to look at a truth table to determine if two *specific* sentences are equivalent. Clearly $1+1=3$ and $10<5$ are logically equivalent since they are both false. The fact they have nothing to do with each other is irrelevant.

De Morgan's Laws

An example of important equivalent sentences are **De Morgan's Laws**[6] **,** which state:

De Morgan's Laws
$\sim(P \vee Q) \equiv \sim P \wedge \sim Q$
$\sim(P \wedge Q) \equiv \sim P \vee \sim Q$

We can verify De Morgan's laws by making truth tables for each side of the equivalences. For the top De Morgan Law above, we have

		(1)	(2)	(3)	(4)	(5)
P	Q	$P \vee Q$	$\sim(P \vee Q)$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$
T	T	T	**F**	F	F	**F**
T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

same truth values

Verification of De Morgan's law

Table 8

> **Interesting Note:** Although sentential logic captures the truth or falsity of simple sentences, it will never replace natural languages. Consider two lines from T. S. Eliot's Love Song of J. Alfred Prufrock:

[6] Augustus De Morgan (1806-1871) was an Indian-born British mathematician and logician who formulated what we now call De Morgan's Laws. He was the first person to make the idea of mathematical induction rigorous.

In the room the women come and go
Talking of Michelangelo.

As logicians, we might let

- WC = women come into the room
- WL = women leave the room
- WT = women talk of Michelangelo

and restate Eliot's poem as $WC \wedge WL \wedge WT$. This conjunction captures the underlying facts of the situation, but the essence of the poem is undoubtedly lost.

Tautology

Definition: A **tautology** is a sentence (normally compound) that is true for all truth values of its components[7].

For example, $P \vee \sim P$, called the **Law of the Excluded Middle**, is a tautology since it is true regardless of the truth of P as the following truth table shows.

P	$\sim P$	$P \vee \sim P$
T	F	**T**
F	T	**T**

Verification that $P \vee \sim P$ is a tautology
Table 9

The Law of the Excluded Middle says everything is either true or false and nothing else. There is no grey area in a world obeying this law. This seems a trivial idea, but this principle is the key step, when later in this book, we resort to proofs by contradiction.

Example 5: Verifying a Tautology Show that

$$(P \wedge Q) \vee (\sim P \vee \sim Q)$$

is a tautology.

Solution: All truth values of $(P \wedge Q) \vee (\sim P \vee \sim Q)$ in the following truth table are T.

[7] In natural language a tautology is often though of as a sentence that says the same thing twice with different words.

P	Q	$P \wedge Q$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$	$(P \wedge Q) \vee (\sim P \vee \sim Q)$
T	T	T	F	F	F	**T**
T	F	F	F	T	T	**T**
F	T	F	T	F	T	**T**
F	F	F	T	T	T	**T**

Verification of a tautology
Table 10

The opposite of a tautology is a contradiction.

Definition: A **contradiction** is a sentence that is false for every truth value of its components.

An example of a trivial contradiction is $P \wedge \sim P$, as in "It is raining and it is not raining." No matter what the truth value of *P,* the truth value of $P \wedge \sim P$ is false. Many contradictions are obvious while others are not so easy to determine. Is the sentence $(P \wedge \sim Q) \wedge (Q \wedge R)$ false for all truth values of $P, Q,$ and R? The answer is yes and with some thought you can convince yourself of this without resorting to a truth table.

			(1)	(2)	(3)	(4)
P	Q	R	$\sim Q$	$P \wedge \sim Q$	$Q \wedge R$	$(P \wedge \sim Q) \wedge (Q \wedge R)$
T	T	T	F	F	T	**F**
T	T	F	F	F	F	**F**
T	F	T	T	T	F	**F**
T	F	F	T	T	F	**F**
F	T	T	F	F	T	**F**
F	T	F	F	F	F	**F**
F	F	T	T	F	F	**F**
F	F	F	T	F	F	**F**

Verification of a contradiction
Table 11

Logical Sentences from Truth Tables: DNF and CNF

Until now, we have found truth tables associated with various logical sentences. It is possible to go backwards from a truth table to a logical expression. For example, consider the truth table in Table 12.

P	Q	R	?
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

Typical truth table for three variables P, Q, R
Table 12

To find a logical sentence that yields the given truth table. There are two such equivalent logical expressions, called disjunctive and conjunctive normal forms

A logical expression is in **disjunctive normal** form (DNF) if it is a disjunction $(\vee)$ of conjunctions $(\wedge)$, whereas a logical expression is in **conjunctive normal** form if it is a conjunction of disjunctions. Simple examples are the following:

Example of disjunctive normal form: $(P \wedge Q) \vee (R \wedge S)$

Example of conjunctive normal form: $(P \vee Q) \wedge (R \vee S)$

To find the DNF for the truth table in Figure 12, we look at the rows where the value of the truth table is **TRUE,** which are rows 1, 4, 5, and 7 and then creating formulas that yield a T. Doing this, we see

- row 1 is true when $P \wedge Q \wedge R$
- row 4 is true when $P \wedge \sim Q \wedge \sim R$
- row 5 is true when $\sim P \wedge Q \wedge R$
- row 7 is true when $\sim P \wedge \sim Q \wedge R$

which gives us the disjunctive normal form for the truth table in Figure 12:

$$\text{DNF: } (P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R) \vee (\sim P \wedge \sim Q \wedge R)$$

.

To find the conjunctive normal form, we look at the rows where the truth table is **FALSE**, or rows 2, 3, 6 and 8, and from this create the formula that says, how we get this F? This will happen when the following are satisfied:

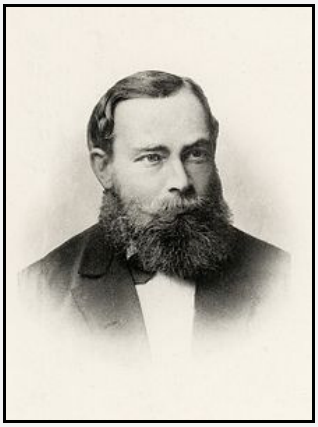
- row 2 is false when $\sim P \vee \sim Q \vee R$
- row 3 is false when $\sim P \vee Q \vee \sim R$
- row 6 is false when $P \vee \sim Q \vee R$
- row 8 is false when $P \vee Q \vee R$

which yields the conjunctive normal form for the truth table in Figure 12:

$$\text{CNF: } (\sim P \vee \sim Q \vee R) \wedge (\sim P \vee Q \vee \sim R) \wedge (P \vee \sim Q \vee R) \wedge (P \vee Q \vee R)$$

You can verify both of these forms are logically equivalent and create the same truth table in Figure 12. Conjunctive and disjunctive normal forms are important in circuit design where designs of integrated circuits are based on Boolean functions.

Historical Note: The first systematic discussion of sentential logic comes from the the German logician Gottried Frege (1848-1925) in the work *Begriffscrift*. Many consider him the greatest logician of the 19th century.



Problems

1. **Simple Sentences** Which of the following are sentences?

a) WE TAKE YOUR BAGS AND SEND THEM IN ALL DIRECTIONS (posted at an airline ticket counter) YES
b) The Riemann hypothesis is still unsolved. YES
c) The Battle of Hastings was fought in 1492. YES
d) The constant π is an algebraic number. YES
e) DROP YOUR TROUSERS HERE FOR THE BEST RESULT. (sign posted at a dry cleaners) NO
f) The constant π is a transcendental number. YES
g) I am a monkey's uncle. YES
h) Never again! YES
i) ABSOLUTELY NO SWIMMING YES
j) $2+5=1$ YES

k) e is an irrational number YES
l) It is not true that 1 + 1 = 2. YES
m) This sentence is false. NO
n) $\int_0^1 x^2 dx = 0$ YES $x^3 + 1 > 0$ NO
o) Digit 0 doesn't appear in π. YES
p) We sit on the porch watching cows playing Scrabble. YES
q) The Chicago Cubs will win the World Series this year YES
r) To be or not to be, that is the question. YES
s) A woman, without her man, is nothing. YES
t) A woman, without her, man is nothing. YES
u) I think, therefore I am. YES
v) All generalizations are false, including this one. NO
w) These pretzels are making me thirsty. YES
x) The 5th order polynomial equation hasn't been solved, YES
y) Who was Niels Abel? **NO**

2. **Truth Tables** For simple sentences P, Q, R make the truth table for the compound sentences.

a) $P \vee \sim P$
b) $P \wedge \sim P$
c) $P \vee \sim Q$
d) $\sim P \vee \sim Q$
e) $\sim (P \vee Q)$
f) $(P \vee Q) \wedge R$
g) $P \wedge (Q \vee R)$
h) $P \vee (Q \wedge R)$
i) $(P \vee Q) \vee (\sim P \vee \sim Q)$
j) $(P \vee Q) \wedge (Q \vee R)$
k) $(P \vee Q) \vee (Q \wedge R)$

3. **True or False** Let

- P be the sentence "$4 > 2$",
- Q be the sentence "$1 + 2 = 3$"
- R be the sentence "$5 + 2 = 9$"

What is the truth value of the following sentences?

a) $P \wedge \sim Q$ Ans: F
b) $\sim (P \wedge Q)$ Ans: F
c) $\sim P \wedge \sim Q$ Ans: F
d) $\sim (P \vee Q)$ Ans: F
e) $P \wedge Q$ Ans: T

4. **Tautologies and Contradictions** Suppose P is a tautology and Q a contradiction. Tell whether the following is a tautology, a contradiction, or neither. Verify your conclusion.

a) $P \wedge Q$ Ans: CONTRADICTION
b) $P \vee Q$ Ans: TAUTOLOGY
c) $Q \vee \sim Q$ Ans: TAUTOLOGY
d) $P \wedge \sim P$ Ans: CONTRADICTION
e) $P \wedge (Q \vee \sim P)$ Ans: CONTRADICTION
f) $P \vee (Q \wedge \sim P)$ Ans: TAUTOLOGY

5. **In Plain English** In English, fill in the blanks in the table.

The sentence	is TRUE when	is FALSE when
P		
$\sim P$		
$P \vee Q$		
$P \wedge Q$		

6. **Denial of Sentences** State the negation of each of the following sentences.
a) π is a rational number.
b) 317 is a prime number.
c) The function $f(x) = x^2 + 1$ has exactly one minimum.
d) It will be cold and rainy tomorrow.
e) It will be either cold or rainy tomorrow.
f) It is not true that I am shiftless and lazy.
g) It is not true that I am either lazy or shiftless.
h) 2 is the only even prime number.

7. **Exclusive OR** In natural language the word "or" requires a certain amount of care. In this lesson we defined the word "or" in the inclusive sense, meaning one or the other or both. The **exclusive or** is slightly different; it means one or the other but *not both*

a) Make a truth table for the exclusive or (denote it by $\oplus$)
b) Verify that $P \oplus Q \equiv (P \vee Q) \wedge \sim (P \wedge Q)$.

8. **Prove or Disprove** Prove or disprove the statement

$$P \oplus Q \equiv (P \wedge \sim Q) \vee (Q \wedge \sim P)$$

where $P \oplus Q$ denotes the **exclusive or**, meaning P or Q but not both.

9. **Alternate Forms for Truth Tables** Truth tables for logical disjunction and conjunction are analogous to addition, multiplication, and negation in a Boolean algebra, where 1 is defined as truth, 0 is taken to be false, and minus as negation. Verify the following statements in Boolean algebra and find their logical equivalents in the sentential calculus.

		P	Q
	×	0	1
P	0	0	0
Q	1	0	1

		P	Q
	+	0	1
P	0	0	1
Q	1	1	1

P	0	1
$\sim P$	1	0

a) $1+(-1)=1$
b) $1\times(-1)=0$
c) $-(1+0)=0\times1$
d) $-(0\times1)=1+0$
e) $1\times(1+0)=(1\times1)+(1\times0)$
f) $1+(1\times0)=(1+1)\times(1+0)$

10. **Logical AND for IP Addresses** A typical IP address for a network on the internet is a 4-dotted string of 8 binary numbers (0s and 1s), which in decimal and binary form might be

$$193.170.6.1 = 11000001.10101010.00000110.00000001$$

The network might consist of 10 computers connected to a router, where each computer in the network is assigned a *subnet mask*. If the subnet mask of one of the computers is

11111111.11111111.11111111.00000000

then the IP of this computer is the logical AND of the network IP and this subnet mask, which is the same as logical binary multiplication.. What is the IP address of this computer using binary arithmetic $0 \cdot 0 = 1,\ 0 \cdot 1 = 0,\ 1 \cdot 0 = 0,\ 1 \cdot 1 = 1$?

Show that this sentence is a tautology.

11. **Disjunctive and Conjunctive Normal Forms** Express the following truth tables in disjunctive and conjunctive normal forms.

a)

P	Q	Logical function
T	T	T
T	F	T
F	T	T
F	F	F

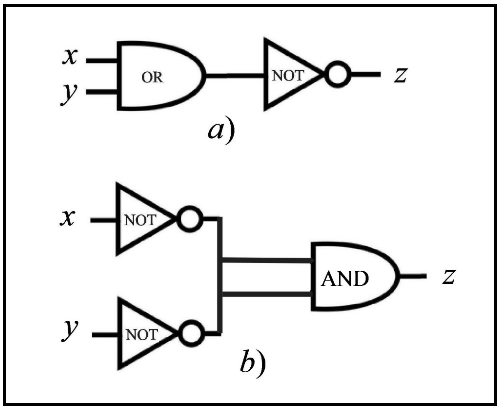
b)

P	Q	Logical function
T	T	T
T	F	F
F	T	F
F	F	T

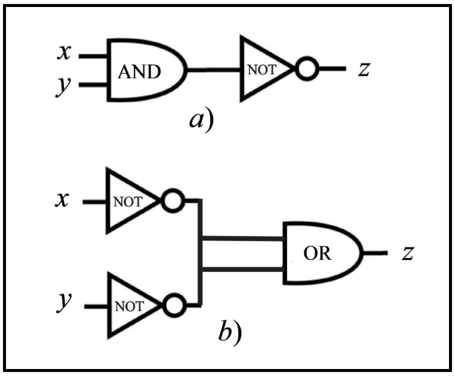
c)

P	Q	R	Logical function
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

12. **Digital Logical Circuits I** Find the truth table for each of the following digital logical circuits to prove they are equivalent. What logical law does the equivalence of these circuits represent? The individual electronic components are self-explanatory.



13. **Digital Logical Circuits II** Find the truth table for each of the following digital logical circuits to prove they are equivalent. What logical law does the equivalence of these circuits represent? The individual electronic components are self-explanatory.



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