Section 1.2 Conditional and Biconditional Connectives

Purpose of Lesson: To introduce the **conditional** and **biconditional** connectives along with various equivalent forms of the conditional sentence, such as the **converse**, **inverse**, and **contrapositive**. These various forms of logical statements play an important role when proving theorems.

The Conditional Sentence

The **conditional sentence** (or **implication**), is a statement of the form

"if P then Q"

which is fundamental in mathematics. From a purely logical point of view, conditional sentences do not necessarily imply a cause and effect between P and Q although they generally do. For example, from a pure logical point of view the conditional sentence

If $1+1=3$*, then pigs fly.*

is a true conditional sentence, although there is no relation between the component parts. A more useful implication is

If a positive integer n *is composite, then* n *has a prime divisor less than or equal to* $\sqrt{n}$

which is a true statement and provides a cause and effect between its components. The reader has seen conditional sentences in Euclidean geometry where much of the subject is explained through implications of this type. The sentence, "If a polygon has three sides, then it is a triangle," is a conditional sentence.

Historical Note: The idea of enumerating all possible truth values in tables we now call "truth tables" seems to have been rediscovered several times throughout history. That said, the philosopher and logician, Ludwig Wittgenstein and Emil Post, respectively used them regularly in the late 1800s where Wittgenstein labeled them "truth tables." Other early researchers who used and made contributions to "symbolic logic" are English logician Bertrand Russell and American logician Benjamin Peirce. Peirce's 1885 paper "*On the Algebra of Logic: A Contribution to the Philosophy of Notation*," published in the *American Journal of Mathematics* includes an example of a truth table for the conditional.

Conditional Sentence : If P and Q are sentences, then the **conditional sentence** "if P then Q" is denoted symbolically by

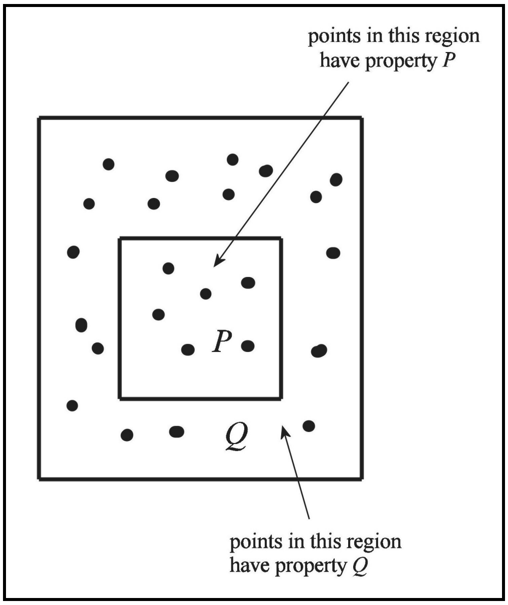
$$P \Rightarrow Q$$

and whose truth values are defined by the truth table:

P	Q	$P \Rightarrow Q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Note that a conditional sentence is false when $T \Rightarrow F$, otherwise it is true. The sentence P is called the **hypothesis** (or **assumption** or **premise**) of the conditional sentence (or implication) and Q is called the **conclusion** (or **consequent**[1]).

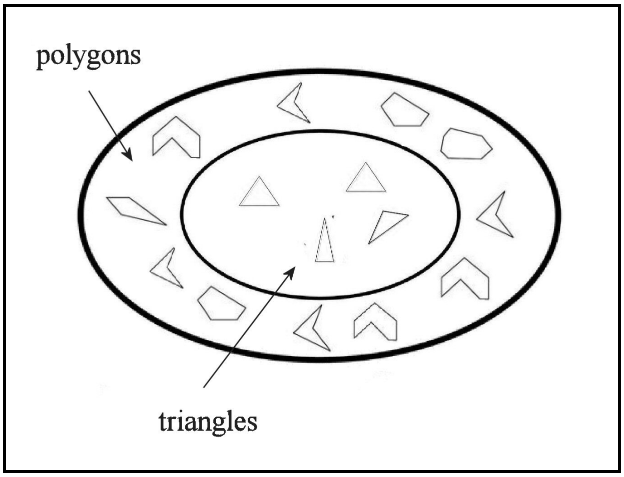
The conditional statement $P \Rightarrow Q$ can be visualized by the **Euler** (or **Venn**) diagram as drawn in Figure 1.



Euler diagram for $P \Rightarrow Q$
Figure 1

[1] In pure logical systems P and Q are generally called the antecedent and consequent, respectively. In mathematics, they are more likely to be called the **hypothesis** and **conclusion**.

For example, all polygons are triangles which we illustrate by the diagram in Figure 2.



If a geometric figure is a triangle, then it is a polygon
Figure 2

Example 1: Conditional Sentences The following sentences are conditional sentences. Are they true or false?

i) If f is a real-valued differentiable function, then f is continuous. TRUE

ii) If N is an even number greater than 2, then N is the sum of two primes. (You get an A for the course if you can prove this statement. Just slide your solution under your professor's door.)

iii) If a and b are the lengths of the legs of a right triangle, and c is the length of the hypotenuse, then $c^2 = a^2 + b^2$. TRUE

Understanding the Conditional Sentence:

The conditional sentence "if P then Q" is best understood as a promise, where if the promise is kept, the conditional sentence is true, otherwise the sentence is false. As an illustration suppose your professor makes you the promise:

If pigs fly, then you will receive an A for the course.

The proposition is true since your professor has only promised an *A* if pigs fly, but since they don't, all bets are off. However, if you see a flying pig outside your classroom and your professor gives you a *C*, then your professor has broken the promise and the proposition is false.

The conditional sentence $P \Rightarrow Q$ is often called an **inference**, and we say *P* **implies** *Q*. Another way of stating $P \Rightarrow Q$ is to say *P* is a **sufficient condition** for *Q,* which means the truth of *P* is sufficient for the truth of *Q*. We also say that Q is a **necessary condition** for P, meaning the truth of Q necessarily follows from the truth of P.

Example 2: Necessary Conditions and Sufficient Conditions.

P	Q	Condition
being pregnant	being female	Q is necessary for P
N is an integer	$2N$ is an integer	P is sufficient for Q
life on earth	air	Q is necessary for P
run over by a truck	squashed	P is sufficient for Q

Necessary conditions and sufficient conditions
Table 1

Important Note: A famous conditional statement is due to the French philosopher/mathematician *Rene Decartes* (1591-1650) who once said "*Cogito ergo sum*", which means "I think therefore I am." As a conditional sentence it would be "If I think, then I am."

Important Note: Normally, in mathematics when one writes the implication $P \Rightarrow Q$ one assumes the hypothesis is true since it makes no sense to assume false hypotheses. In fact, if we assumed a false hypothesis the implication would be true regardless of the truth value of Q.

Converse, Inverse, and the Contrapositive

The implication $P \Rightarrow Q$ gives rise to three related implications, one is equivalent to the implication, the others are not.

Implication	Converse	Inverse	Contrapositive
$P \Rightarrow Q$	$Q \Rightarrow P$	$\sim P \Rightarrow \sim Q$	$\sim Q \Rightarrow \sim P$

It is not difficult to show by truth tables

converse: $P \Rightarrow Q \not\equiv Q \Rightarrow P$

inverse: $P \Rightarrow Q \not\equiv \sim P \Rightarrow \sim Q$

contrapositive: $P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P$

Law of the Syllogism

A fundamental principle of logic, called the **law of the syllogism**, states:

"if P implies Q, and Q implies R, then P implies R"

is equivalent to the compound conditional

$$\left[(P \Rightarrow Q) \wedge (Q \Rightarrow R)\right] \Rightarrow P \Rightarrow R$$

which is a tautology. We can verify this by noting all T's in column (5) of the table in Table 2.

			(1)	(2)	(3)	(4)	**(5)**
P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$P \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	T	F	T	T	F	**T**
T	F	F	F	T	F	F	**T**
F	T	T	T	T	T	T	**T**
F	T	F	T	F	T	F	**T**
F	F	T	T	T	T	T	**T**
F	F	F	T	T	T	T	**T**

Truth table verification of the syllogism
Table 2

Well-Formed Sentences The statement $P \vee Q \wedge R$ is not what we call **well-formed** since its meaning is unclear. One would have to include parentheses to tell which of the (not equivalent) well-formed sentences $(P \vee Q) \wedge R$ or $P \vee (Q \wedge R)$ is intended. Well formed sentences are similar to "well-formed sentences" in English: a capital letter at the start, a period at the end, and all the other rules of grammar in between.

A Useful Equivalence for the Implication

The implication $P \Rightarrow Q$ is true if either P is false or Q is true. Hence, we have the logical equivalence

$$P \Rightarrow Q \equiv \sim P \vee Q$$

which we verify by means of the truth table in Table 3.

P	Q	$P \Rightarrow Q$	$\sim P$	$\sim P \vee Q$
T	T	**T**	F	**T**
T	F	**F**	F	**F**
F	T	**T**	T	**T**
F	F	**T**	T	**T**

Equivalence of $P \Rightarrow Q \equiv \sim P \vee Q$
Table 3

Note too this means the negation of the implication $P \Rightarrow Q$ is

$$\sim (P \Rightarrow Q) \equiv \sim (\sim P \vee Q) \equiv P \wedge \sim Q$$

In other words, an implication is false when the premise is true and the conclusion false.

The Biconditional

Statements of the form

"*P* if and only if *Q*"

are fundamental in mathematics. This leads us to the following definition.

Definition: If P and Q are sentences, then the **biconditional** sentence "P if and only if Q" is denoted by

$$P \Leftrightarrow Q$$

whose truth values are given by the truth table

P	Q	$P \Leftrightarrow Q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

$P \Leftrightarrow Q$ is often read as "P **if and only if** Q" or P iff Q for shorthand. Another phrasing of $P \Leftrightarrow Q$ is P is a **necessary and sufficient condition** for Q. In other words, $P \Leftrightarrow Q$ is true if and only if P and Q have the same truth value.

Example 3: Biconditional Equivalent to two Implications

Show that the biconditional $P \Leftrightarrow Q$ is equivalent to

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P).$$

Solution The truth values in the truth table under (3) and (4) in Table 4 are the same.

		(1)	(2)	(3)	(4)
P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	$P \Leftrightarrow Q$
T	T	T	T	**T**	**T**
T	F	F	T	**F**	**F**
F	T	T	F	**F**	**F**
F	F	T	T	**T**	**T**

Equivalence of $(P \Rightarrow Q) \wedge (Q \Leftarrow P) \equiv P \Leftrightarrow Q$

Table 4

Warning: Be careful not to confuse the biconditional connective $P \Leftrightarrow Q$ with the equivalence $P \equiv Q$. The equivalence relation says P and Q are *logical equivalent compound* sentences. The biconditional $P \Leftrightarrow Q$ does *not* necessarily mean P and Q have the same truth values since the biconditional can be false. However, when the biconditional $P \Leftrightarrow Q$ is true, then P and Q have the same truth values.

Example 4: Truth Values of Biconditional Sentences

Biconditional	Truth Value
$1+2=5$ if and only if $1-3=4$	True
$3+5=8$ if and only if $3\times 4=12$	True
$1+2=3$ if and only if $(a+b)^2=a^2+b^2$	False
$\pi=22/7$ if and only if $6/3=2$	False

Important Note: Greek philosophers called **Modus Ponens** the valid argument that if P is true and if $P \Rightarrow Q$ is true, then Q is true, which in sentential logic notation is $\left[P \wedge (P \Rightarrow Q)\right] \Rightarrow Q$.

Important Note: The origin of the "iff" notation, meaning "if and only if" first appeared in print in 1955 in the text *General Topology* by John Kelly although its invention is generally credited to the Hungarian/American Paul Halmos.

Problems

Working Definitions: The following definitions are needed in some problems in this and in following sections.

- An integer n **divides** an integer m (and we write $n|m$) if there exists an integer q such that $m=n\times q$. If n does **not divide** m, we write $\nmid$.
- An integer n is **even** if there exists an integer k such that $n=2k$.
- An integer n is **odd** if there exists an integer k such that $n=2k+1$.
- A natural number $\mathbb{N}=\{1,2,3,...\}$ is **prime** if it is only divisible by 1 and itself.

1.**True or False** Identify the assumption and conclusion in the following conditional sentences and tell if the implication is true or false.

a) If pigs fly, then I am richer than Bill Gates.
b) If a person got the plague in the 17th century, they die.
c) If you miss class over 75% of the time, you are in trouble.
d) If x is a prime number then x^2 is prime too.
e) If x and y are prime numbers, then so is $x+y$.
f) If the determinant of a matrix is nonzero, the matrix has an inverse.
g) If f is a 1-1 function, then f has an inverse.

2. **Contrapositive** Write the contrapositive for the conditional sentences in Problem 1.

3. **True or False** Let P be the sentence "$4>6$", Q the sentence "$1+1=2$", and R the sentence "$1+1=3$".What is the truth value of the following sentences?

a) $P \wedge \sim Q$ Ans: F
b) $\sim(P \wedge Q)$ Ans: T
c) $\sim(P \vee Q)$ Ans: F
d) $\sim P \wedge \sim Q$ Ans: T
e) $P \wedge Q$ Ans: F
f) $P \Rightarrow Q$ Ans: T
g) $Q \Leftrightarrow R$ Ans: F
h) $P \Rightarrow (Q \Rightarrow R)$ Ans: T
i) $(P \Rightarrow Q) \Rightarrow R$ Ans: F
j) $(R \vee Q \vee R) \Leftrightarrow (P \wedge Q \wedge R)$ Ans: F

4. **True or False** Let P be the sentence “Jerry is richer than Mary,” Q is the sentence “Jerry is taller than Mary,” and R is the sentence “Mary is taller than Jerry.” For the following sentences, what can you conclude about Jerry and Mary if the given sentence is true. Express the information in a convenient form.

a) $P \vee Q$
b) $P \wedge Q$
c) $\sim P \vee Q$
d) $Q \wedge R$
e) $\sim Q \wedge \sim R$

f) $P \wedge (P \Rightarrow Q)$
g) $P \Leftrightarrow (Q \vee R)$
h) $Q \wedge (P \Rightarrow R)$
i) $P \vee Q \vee R$
j) $P \vee (Q \wedge R)$

5. **Truth Tables** Construct truth tables to verify the following equivalences.

a) $(P \Leftrightarrow Q) \equiv (\sim P \Leftrightarrow \sim Q)$
b) $[\sim (P \Leftrightarrow Q)] \equiv [(P \wedge \sim Q) \vee (\sim P \wedge Q)]$
c) $(P \Rightarrow Q) \equiv (\sim P \vee Q)$

6. **Conditional Sentences** Translate the given sentences in English to the form $P \Rightarrow Q$.

a) Unless you study, you won't get a good grade.
b) "Do you like it? It's yours."
c) Get out or I'll call the cops.
d) Anyone who doesn't study deserves to flunk.
e) Criticize her and she will slap you.
f) With his toupee on, the professor looks younger.

7. **In Plain English** Without making a truth table, say why the following implications are true.

a) $[(P \vee Q) \wedge \sim P] \Rightarrow Q$
b) $[P \wedge (Q \wedge \sim Q)] \Rightarrow \sim P$
c) $(P \vee Q) \Rightarrow (\sim P \Rightarrow Q)$

8. **AND and OR** For P, Q and R verify the distributive laws

a) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
b) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

9. **Inverse, Converse, and Contrapositive** One of the following sentences has the same meaning as $P \Rightarrow Q$. Which one is it?

$$\textit{inverse}: \sim P \Rightarrow \sim Q$$
$$\textit{converse}: Q \Rightarrow P$$
$$\textit{contrapositive}: \sim Q \Rightarrow \sim P$$

For the two sentences not equivalent to $P \Rightarrow Q$ find an example illustrating this fact.

10. **True or False** Is the following statement a tautology, a contradiction, or neither?

$$\left[(P \Rightarrow Q) \wedge Q\right] \Rightarrow P$$

11. Show the equivalence of the following implications.

a) $P \Rightarrow Q$ (direct form of an implication)
b) $\sim Q \Rightarrow \sim P$ (contrapositive form)
c) $(P \wedge \sim Q) \Rightarrow \sim P$ (proof by contradiction)
d) $(P \wedge \sim Q) \Rightarrow Q$ (proof by contradiction)
e) $(P \wedge \sim Q) \Rightarrow R \wedge \sim R$ (*reduction ad absurdum*)

12. **Hmmmmmm** Is the statement

$$(P \vee Q) \Leftrightarrow (P \vee \sim Q)$$

true for all truth values of P and Q, or is it false for all values, or is it sometimes true and sometimes false?

13. **Interesting Biconditional** Is the statement

$$(P \vee Q) \Leftrightarrow (\sim P \vee \sim Q)$$

true for all truth values of P and Q, or is it false for all values, or is it sometimes true and sometimes false?

14. Find the negation of the following sentences.

a) $(P \vee Q) \wedge R$
b) $(P \vee Q) \wedge (R \vee S)$
c) $(\sim P \vee Q) \wedge R$

15. Give, if possible, an example of a true conditional for which

a) the contrapositive is true
b) the contrapositive is false

c) the converse is true
d) the converse is false

16. The **inverse** of the implication $P \Rightarrow Q$ is $\sim P \Rightarrow \sim Q$.

a) Prove or disprove that an implication and its inverse are equivalent.

b) What are the truth values of P and Q for which an implication and its inverse are both true?

c) What are the truth values of P and Q for which the implication and its inverse are both false?

17. For the sentence

"If N is an integer, then 2N is an even integer"

write the converse, contrapositive, and inverse sentences.

18. Let P, Q, and R be sentences. Show

a) $P \Rightarrow (Q \Leftrightarrow R)$ requires paranthesis

b) $(P \wedge Q) \vee R$ requires paranthesis

c) $(\sim P \vee Q) \Rightarrow R$ may be written $\sim P \vee Q \Rightarrow R$

19. **Challenge** Rewrite the sentence

$$P \Rightarrow (Q \Rightarrow R)$$

in an equivalent form in which the symbol "$\Rightarrow$" does not occur.

20. **Non Obvious Statement** The statement

$$P \Rightarrow (Q \Rightarrow P)$$

can be read "If P is true, then P follows from any Q" Is this a tautology, contradiction, or does its truth value depend on the truth or falsity of P and Q?

21. **Another Non Obvious Statement** The statement

$$(Q \Rightarrow P) \vee (P \Rightarrow Q)$$

can be read "For any two sentences P and Q, it is always true that

P implies Q or Q implies P"

Is this a tautology, contradiction, or does its true value depend on the truth or falsity of P and Q?

22. **Three-Valued Logic** Two-valued (T and F) truth tables were basic in logic until 1921 when the Polish logician Jan Lukasiewicz (1878-1956) and American logician Emil Post (1897-1954) introduced n-valued logical systems where n is any integer greater than 1. For example, sentences in a three-valued logic might have values True, False, and Unknown. Three-value logic is useful in computer science in database work. The truth tables for the AND, OR, and NOT connectives are as follows:

A	***B***	***A* OR *B***	***A* AND *B***	**NOT *A***
True	True	True	True	False
True	Unknown	True	Unknown	False
True	False	True	False	False
Unknown	True	True	Unknown	Unknown
Unknown	Unknown	Unknown	Unknown	Unknown
Unknown	False	Unknown	False	Unknown
False	True	True	False	True
False	Unknown	Unknown	False	True
False	False	False	False	True

From these connectives, derive the connectives for the conditional and biconditional connectives.

23. **Anyone for Modus Ponens**[2] and **Modus Tollens?**[3] are systematic ways of making logical arguments that takes the form

Modus Ponens	Modus Tollens
If P then Q	If P then Q
P	$\sim Q$
Therefore Q	Therefore $\sim P$

Show that Modus Ponens and Modus Tollens are both tautologies.

24. **Interesting** Are the following two statements equivalent?

$$P \wedge (Q \Rightarrow R)$$
$$(P \wedge Q) \Rightarrow R$$

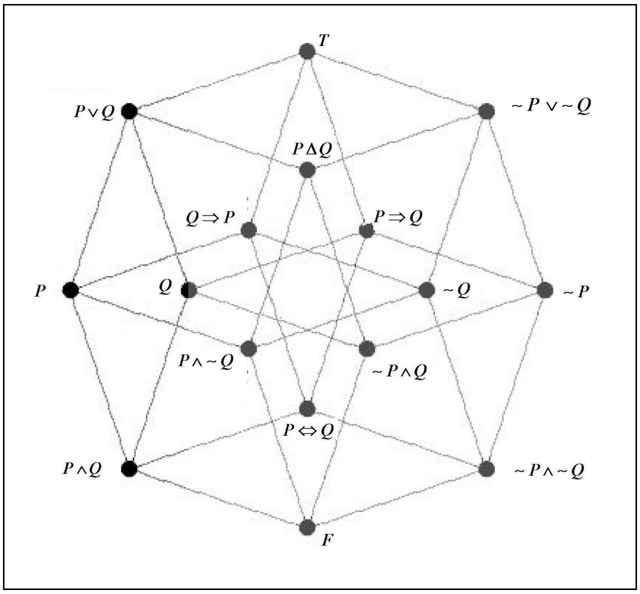
25. **Sixteen Logical Functions of Two Variables** The diagram below shows the totality of sixteen relations between two logical variables. One expression

[2] Latin: *mode that affirms.*
[3] Latin *mode that denies.*

can be proven from another if it lies on an upward path from the first. For example

$$(P \wedge Q) \Rightarrow Q \Rightarrow (P \Rightarrow Q).$$

Verify a few of these implications using truth tables. The compound sentence $P \Delta Q$ refers to the exclusive OR, which means either P or Q is true but not both.



26. **Professor Snarf's Birthday** Professor Snarf tells Mary and Dave his birthday is one of the days in the following table. He then tells Mary the month of his birthday is March, and tells Dave the day of the month of his birthday is Day 1, but tells them not to pass this information to the other. Professor Snarf then tells them if they can deduce his birthday, they will each receive an A in their logic course, knowing the problem is impossible. However, Professor Snarf does not realize the cleverness of Mary and Dave.

	Day					
	1	2	3	4	5	6
Jan			Jan 3	Jan 4		Jan 6
Feb			Feb 3		Feb 5	
March	March 1			March 4		
April	April 1	April 2				April 6

The conversation between Mary and Dave goes like this:

- **Statement 1:** Mary says, "If I don't know the answer, then neither do you."
- **Statement 2**: Dave says "I didn't know the answer before, but I do now and so do you."
- **Statement 3**: Mary says: "I know the answer too."

whereupon they correctly tell Professor Snarf his birthday, and he grudgenly gives them their promised grades. What was the correct answer Mary and Dave gave Professor Snarf and how did they determine it?

Ans: Mary, knowing the month to be March, states if she doesn't know Professor Snarf's birthday neither does Dave, which is her way of telling Dave the birthday is not in February or April, since if it were Dave would know the birthday (look at the table). So Mary and Dave have reduced the number of possible birthdays from 10 to 5, which are illustrated in the table below.

	Day					
	1	2	3	4	5	6
Jan		[redacted]	Jan 3	Jan 4	[redacted]	Jan 6
[redacted]						
March	March 1	[redacted]		March 4	[redacted]	
[redacted]						

Dave, knowing the birthday is Day 1, now knows Professor Snarf's birthday is on March 1, so he says he knows Professor Snarf's birthday and so does Mary. Mary, hearing Dave say he knows the birthday then knows the birthday is not Day 4, and hence is on Day 1, so she also concludes Professor Snarf's birthday is March 1.

Day						
	1	2	3	4	5	6
Jan		[redacted]	Jan 3	[redacted]		Jan 6
[redacted]						
March	March1	[redacted]		[redacted]		
[redacted]						

ΓΣΘΨΞΠΩ