Section 1.3 Predicate Logic

Purpose of Section: We now introduces **predicate logic** (or **first-order logic**), which is the "language" of mathematics. We will see how predicate logic extends the language of sentential calculus introduced in the previous sections by adding **universal** and **existential quantifiers**, **logical functions** and **variables**.

Introduction

Although sentential logic introduced in Sections 1.1 and 1.2 is sufficient to get you through your daily activities, it is not sufficient for higher mathematics. This observation was realized in the late 1800s by the German logician Gottlob Frege who proposed a more extensive language than the simple sentential logic of P and Q and their connectives $\wedge, \vee, . \sim, \Rightarrow, \Leftrightarrow$.

What Frege introduced is called **predicate logic**[1] (or **first-order logic**), which consists of the logical connectives of sentential logic, plus the extra ingredients: **quantifiers**, **variables**, and logical functions called **predicates**. Predicate logic allows one to express concepts about *collections* of objects, such as when we say

> *"for any real number x there exists a real number y such that $x < y$ "*

which we could not express in the simpler language of sentential logic.

Existential and Universal Quantifiers

Two phrases one often hears again and again in mathematics are "*for all*," and "*there exists*." These expressions are called **quantifiers** and are necessary for the precise description of mathematical concepts. The meaning of an expression like $x < y$ in itself is unclear until we define the meaning and extent of x and y. This leads us to the two basic quantifiers of predicate logic. The **universal quantifier**, meaning "for all" and denoted by $\forall$ (upside down A), and the **existential quantifier,** meaning "there exists," and denoted by $\exists$ (backwards E). Inherent in the use of quantifiers is the concept of a **universe** set, often the real numbers, integers, natural numbers, and so on.

[1] Predicate logic is also called **first-order logic** in contrast to sentential calculus, which is sometimes called **zero-order logic**, studied in Sections 1.1 and 1.2.

Quantifiers of Predicate Logic Let U be the universe under consideration.

- **Universal Quantifier**: The proposition $(\forall x \in U)\ P(x)$ means "for all (or any) x in U, the proposition $P(x)$ is true"
- **Existential Quantifier:** The proposition $(\exists x \in U)\ P(x)$ means "there exists an x in U such that $P(x)$ is true"

Some common number universes in mathematics are the following.[2]

$\mathbb{N} = \{1, 2, 3, \ldots\}$ (natural numbers)
$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ (integers)
$\mathbb{Q} = \{ p/q : p \text{ and } q \text{ are integers with } q \neq 0 \}$ (rational numbers)
$\mathbb{R} = \{\text{real numbers}\}$
$\mathbb{C} = \{\text{complex numbers}\}$

More than One Variable in a Proposition

Propositions in predicate logic often contain more than one variable, which means the proposition must contain more than one quantifier. The following table illustrates propositions with two variables and what it means for the propositions to be true or false. To simplify notation, we assume the universe is known so we do not include it.

Proposition	Proposition is true when	Proposition is false when
$(\forall x)(\forall y)P(x,y)$	For all x and y $P(x,y)$ is true	$P(x,y)$ is false for some x, y.
$(\exists x)(\forall y)P(x,y)$	There is an x such that $P(x,y)$ is true for all y.	For all x, $P(x,y)$ is false for some y.
$(\forall x)(\exists y)P(x,y)$	For all x $P(x,y)$ is true for some y.	There is an x such that $P(x,y)$ is false for all y .
$(\exists x)(\exists y)P(x,y)$	$P(x,y)$ is true for some x, y.	$P(x,y)$ is false for all x, y.

[2]We have gotten ahead of ourselves and introduced a modicum of set notation although we suspect most readers are well familiar with our mild use of sets and set notation.

Example 1: Converting Predicate Logic to English

The following propositions are translated into natural language.

Proposition	English Meaning	Truth Value
$(\forall x \in \mathbb{R})\ (x^2 \geq 0)$	For any real number, its square is nonnegative.	true
$(\exists x \in \mathbb{N})(x \text{ is a prime number})$	There exists a prime number	true
$(\exists n \in \mathbb{N})\ (2 \nmid n)$	There exists at least one odd natural number.	true
$(\exists n \in \mathbb{N})\ (2 \mid n)$	There exists at least one even natural number.	true
$(\forall x \in \mathbb{N})\ (\exists y \in \mathbb{N})\ (x = y + 1)$	For any natural number x, there is a natural number y satisfying $x = y + 1$.	false
$(\forall x \in \mathbb{R})\ (\exists y \in \mathbb{R})\ (x < y)$	For any real number x, there is a real number y greater than x.	true
$(\exists x \in \mathbb{R})\ (\forall y \in \mathbb{R})\ (x < y)$	There exists a real number x such that all real numbers y are greater than x..	false
$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y = 2x)$	For any positive real number x, there is a real number y such that $y = 2x$.	true

Sentences in predicate logic
Table 1

Historical Note: Frege's 1879 seminal work *Begriffsschrift* ("Conceptual Notation") marked the beginning of a new era in logic, which allowed for the quantification of mathematical variables, just in time for the more precise **arithmetization of analysis** of calculus, being carried out in the late 1800s by mathematicians like the German Karl Weierstrass (1815-1897).

Order Matters

Here's a question to ponder. Do the two propositions $(\exists x)(\forall y)P(x, y)$ and $(\forall y)(\exists x)\,P(x, y)$ mean the same thing or does one imply the other or are they completely unrelated? For example, consider the two propositions:

- $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x < y)$
- $(\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(x < y)$

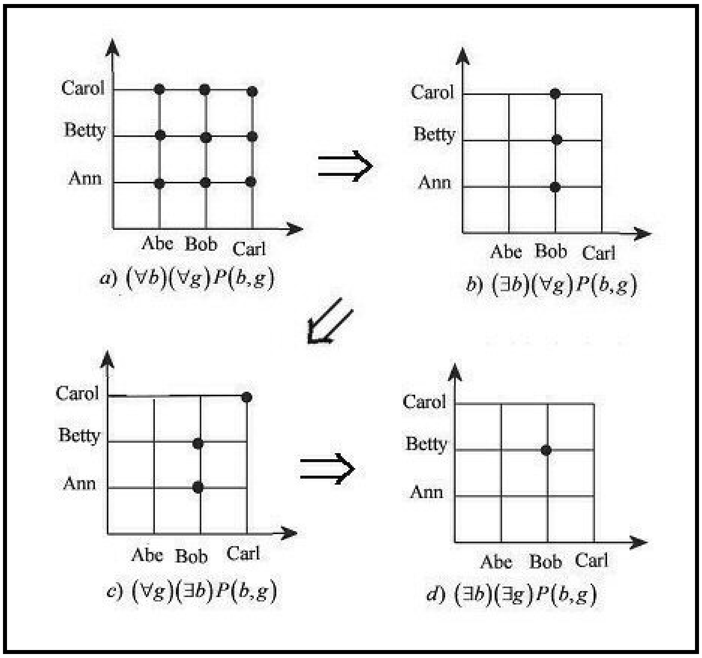
Do they mean the same thing? The answer is no since the proposition on the top is false while the proposition on the bottom is true. To show how the order of the universal and existential quantifiers makes a difference in the meaning of a proposition, we present a simple scenario of a third-grade class consisting of three boys and three girls, members of the sets

$$B = \{\text{Abe, Bob, Carl}\}$$
$$G = \{\text{Ann, Betty, Carol}\}$$

and the predicate

$$P(b,g) = \text{ boy } b \text{ likes girl } g \,.$$

where Figure 1 illustrates visually the relationships between the quantifiers $\forall\forall, \exists\forall, \forall\exists, \exists\exists$. The dot at the intersection of a boy and girl indicates the boy likes the girl. For example, in Figure 1a) every boy likes every girl, while in Figure 1c) this means Bob likes Ann and Betty, and Carl likes Carol.



Universal and Existential Implications
Figure 1

Figure 1 gives visual support for the implications

$$\forall g \forall b \equiv \forall b \forall g \Rightarrow \exists b \forall g \Rightarrow \forall g \exists b \Rightarrow \exists g \exists b \equiv \exists b \exists g$$

which we state explicitly in Figure 2.

Predicate logic	Equivalent English
$(\forall b\in B)(\forall g\in G)P(b,g)$	every boy likes every girl
$\Downarrow$	$\Downarrow$
$(\exists b\in B)(\forall g\in G)P(b,g)$	some boy likes every girl
$\Downarrow$	$\Downarrow$
$(\forall g\in G)(\exists b\in B)P(b,g)$	every girl is liked by some boy
$\Downarrow$	$\Downarrow$
$(\exists g\in B)(\exists g\in G)P(b,g)$	some girl is like by some boy

$$\forall g\forall b \equiv \forall b\forall g \Rightarrow \exists b\forall g \Rightarrow \forall g\exists b \Rightarrow \exists g\exists b \equiv \exists b\exists g$$

Figure 2

Negation of Propositions

In the next few sections when we introduce strategies for proving theorems, it is necessary to know how to negate quantified statements. The following table shows how to negate a few propositions in predicate logic. We assume the universe U is known so we don't include it.

Proposition	Negation of Proposition
$(\forall x)\ P(x)$	$(\exists x)\ [\sim P(x)]$
$(\exists x)\ P(x)$	$(\forall x)[\sim P(x)]$
$(\forall x)(\exists y)\ P(x,y)$	$(\exists x)(\forall y)\ [\sim P(x,y)]$
$(\exists x)(\forall y)\ P(x,y)$	$(\forall x)(\exists y)[\sim P(x,y)]$
$(\forall x)(\forall y)P(x,y)$	$(\exists x)(\exists y)[\sim P(x,y)]$
$(\exists x)(\exists y)P(x,y)$	$(\forall x)(\forall y)[\sim P(x,y)]$

Negation in predicate logic
Table 2

Example 2: Negations State the negation of the following propositions. We don't specify the universe for the variables.

a) $(\forall x)\left[x>0 \Rightarrow (\exists y)(x+y=1)\right]$

b) $(\exists n\in S)(n \text{ is a prime number})$

c) $(\forall x)(\exists y)(xy=10)$

d) $(\exists x)(\forall y)(xy\neq 10)$

Solution

a) $(\exists x)\left[(x>0)\wedge(\forall y)(x+y\neq 1)\right]$

b) $(\forall n\in S)(n \text{ not a prime number})$

c) $(\exists x)(\forall y)(xy\neq 10)$

d) $(\forall x)(\exists y)(xy=10)$

Logicism: The development of predicate logic is generally attributed to the German logician Gottlob Frege (1848-1925), considered to be the most important logician of the 19th century. It was Frege's belief (misguided as it turned out) that all mathematics could be derived from logic. The philosophy that mathematics is a branch of logic and that all mathematical principles are reducible to logical principles is called **logicism**.

Predicate Logic in Analysis Real analysis is an important area of mathematics where detailed precision is critical. Here are a few definitions stated in the language of predicate logic related to sequences $x_n, n=1,2,\ldots$ of real numbers.

- $\lim\limits_{n\to\infty} x_n = 2 \Leftrightarrow (\forall\varepsilon>0)(\exists N\in\mathbb{N})(\forall n>N)(|x_n-2|<\varepsilon)$
- $\lim\limits_{n\to\infty} x_n \neq 2 \Leftrightarrow (\exists\varepsilon>0)(\forall N\in\mathbb{N})(\exists n>N)(|x_n-2|\geq\varepsilon)$
- $\{x_n\}$ is Cauchy $\Leftrightarrow (\forall\varepsilon>0)(\exists N\in\mathbb{N})(\forall m,n>N)(|x_m-x_n|<\varepsilon)$
- .$\{x_n\}$ not Cauchy $\Leftrightarrow (\exists\varepsilon>0)(\forall N\in\mathbb{N})(\exists m,n>N)(|x_m-x_n|\geq\varepsilon)$

Conjunctions and Disjunctions in Predicate Logic

Table 3 shows relations between propositions containing both disjunctions $(\vee)$ and conjunctions $(\wedge)$:

- $(\forall x)(P(x)\wedge Q(x)) \Leftrightarrow (\forall x)P(x)\wedge(\forall x)Q(x)$
- $(\forall x)P(x)\vee(\forall x)Q(x) \Rightarrow (\forall x)(P(x)\vee Q(x))$
- $(\exists x)(P(x)\wedge Q(x)) \Rightarrow (\exists x)P(x)\wedge(\exists x)Q(x)$
- $(\exists x)(P(x)\vee Q(x)) \Leftrightarrow (\exists x)P(x)\vee(\exists x)Q(x)$

Quantifying conjunctions and disjunctions
Table 3

As an aid in understanding these relationships, Professor Snarf has conducted a survey among his students about food preferences. The students respond in one of four ways:

- they like both peanuts and qumquats[3]
- they like only peanuts
- they like only qumquats
- they like neither

If

$$P(s) = \text{ student } s \text{ likes peanuts}$$
$$Q(s) = \text{ student } s \text{ likes qumquats}$$

the statements in Table 3 have the following meanings:

1. $(\forall s)\left[P(s) \wedge Q(s)\right]$ all students like p and q
2. $(\forall s)P(s) \wedge (\forall s)Q(s)$ all students like p and all students like q
3. $(\exists s)\left[P(s) \wedge Q(s)\right]$ some student likes p and q
4. $(\exists s)P(s) \wedge (\exists s)Q(s)$ some student likes p and some likes q
5. $(\exists s)\left[P(s) \vee Q(s)\right]$ some student like p or q
6. $(\exists s)P(s) \vee (\exists s)Q(s)$ some student likes p or some likes q
7. $(\forall s)P(s) \vee (\forall s)Q(s)$ all students like p or all students like q
8. $(\forall s)\left[P(s) \vee Q(s)\right]$ all students like p or q

Figure 3 give a visual aid in understanding the relationships of quantified disjunctions and conjunctions given in Table 2. For example, Figure 3b) illustrates the implication

$$(\forall s)P(s) \vee (\forall s)Q(s) \Rightarrow (\forall s)\left[P(s) \vee Q(s)\right]$$

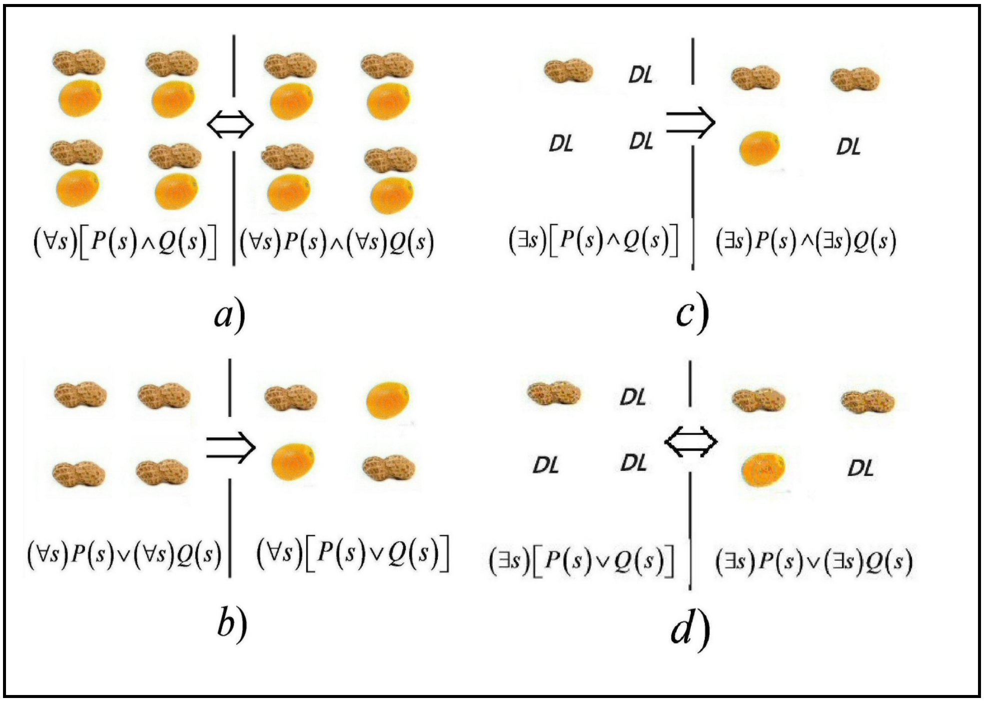
which states that if every student like peanuts or every student likes qumquats, then every student likes peanuts or qumquats. But the converse does not hold inasmuch as

$$(\forall s)\left[P(s) \vee Q(s)\right]$$

means each student likes peanuts or qumquats, but that doesn't imply every student just likes peanuts or every student just likes qumquats, which is what

[3] Author apologizes for the misspelling of kumquats, but the example desperately needed a fruit that started with "q.".

$(\forall s)P(s) \vee (\forall s)Q(s)$ says. The *DL* in Figure 3 means a student *DOESN'T LIKE* either peanuts or qumquats.



Food preferences of four people that match different predicates
Figure 3

No Truth Tables in Predicate Logic The above demonstration involving the 3 boys and 3 girls is by no means a rigorous proof that $\forall\forall \Rightarrow \exists\forall \Rightarrow \forall\exists \Rightarrow \exists\exists$. It is not easy to prove implications in first-order logic since truth tables are not available. For example, although we know the statement $(\forall x \in \mathbb{R})(x^2 \geq 0)$ is true, to make a truth table to verify it would a row for each real number x and a T next to it, showing the proposition true for the given value of x. The problem, of course, is that would require an infinite number of rows!

Historical Note: Although Gottlob Frege is credited with the development of predicate logic, Aristotle anticipated quantifiers 2000 years earlier in his development of syllogisms. The most famous is "All people are mortal. Socrates is a person. Therefore, Socrates is mortal." Also, earlier in the 19th century, the English logician Augustus DeMorgan developed the notion of quantifiers, it was Frege who developed the complete theory of predicate logic as we know it today.

Example 3: All Even or Odd versus All Even or All Odd

Consider the propositions involving an integer $n \in \mathbb{Z}$:

$$O(n) = n \text{ is odd}$$

$$E(n) = n \text{ is even}$$

and compare the following statements.

- $(\forall n \in \mathbb{Z})(E(n)) \vee (\forall n \in \mathbb{Z})(O(n))$
- $(\forall n \in \mathbb{Z})(E(n) \vee O(n))$

Are they equivalent or does one imply the other?

Solution:

The following true implication states "all integers are even or all integers are odd" implies "all integers are even or odd."

$$\left[(\forall n \in \mathbb{Z})(E(n)) \vee (\forall n \in \mathbb{Z})(O(n))\right] \Rightarrow (\forall n \in \mathbb{Z})(E(n) \vee O(n))$$

The assumption in the above implication is false (this is like saying every integer is like 8, -4, 20, .. or like 1, -5, 9, 25,). But a false assumption means a true implication. On the other hand, the converse says if every integer is even or odd[4] which we know to be true, then we conclude a fact we know to be false. Hence, the assumption is true and the conclusion false. Hence, the converse does not hold, meaning

$$(\forall n \in \mathbb{Z})(E(n) \vee O(n)) \not\Rightarrow \left[(\forall n \in \mathbb{Z})(E(n)) \vee (\forall n \in \mathbb{Z})(O(n))\right]$$

Historical Note: In the early 1900s, German mathematician David Hilbert (1862-1943) attempted to *formalize* mathematics within the language of first-order logic by proposing a series of axioms from which all of mathematics would follow. To this end, he attempted to find axioms that would prove or disprove every given theorem. If **Hilbert's Grand Plan** had succeeded, it would put mathematicians out of business and turned them into "deduction robots" turning mathematics into a "turn-the-crank" predicate-logic system. Unfortunately, Hilbert's grand scheme failed due to Gödel's **Incompleteness Theorem** of 1931, which proved no matter what axiom system[5] is chosen, there are always theorems that can never be proven or disproved[6].

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[4] We could prove this, but we will take it as fact.

[5] Hilber't's plan to formalize mathematics can be found online by interested readers.

[6] Then too, mathematics is more than a pure "deductive" discipline. It relies a great deal on intuition and creativity.

Problems

1. Write the following quotes in the symbolic language of predicate logic.

a) Learn from yesterday, live for today, hope for tomorrow.
...Albert Einstein

b) A woman can say more in a sigh than a man in a sermon.
...Arnold Haultain

c) "We all go a little mad sometimes."
...Norman Bates in the movie Psycho (1960)

d) Cowards die many times before their deaths; the valiant never taste of death but once. ...
William Shakespeare

e) All people are mortal.
f) All that glitters is not gold.

2.**Translate to Predicate Logic** Write the following sentences in the symbolic language of predicate logic. The universe of each variable is given in parentheses. For these problems, we use the notation

$\mathbb{Z} =$ integers
$\mathbb{R} =$ real numbers

a) If $a|b$ and $b|c$, then $a|c$, where a,b,c are integers. (Integers)
b) 4 does not divide n^2+2 for any integer (Integers)
c) $x^3+x+1=0$ for some real x (Real numbers)
d) Everybody loves mathematics. (All people)
e) For every positive real number a there exists a real number x that satisfies $e^x = a$. (Real numbers)
f) For every positive real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon$. where a, x are arbitrary real numbers.
g) Everyone always attends class. (All students)
h) The equation $x^2+1=0$ has no solution. (Real numbers)
i) The equation $x^2-2=0$ has no solution. (Rational numbers)

3. **True or False** Which of the following propositions are true in the given universe? The universe is given in parentheses.

a) $(\forall x)(x \le x)$ (Real numbers) Ans: T

b) $(\exists x)(x^2 = 2)$ (Real numbers) Ans: T

c) $(\exists x)(x^2 = 2)$ (Rational numbers)

d) $(\exists x)(x^2 + x + 1 = 0)$ (Real numbers)

e) $(\forall x)[x \equiv 1 (\mathrm{mod}\, 5)]$ (Integers)

f) $(\exists ! x)(e^x = 1)$ (Real numbers)

g) $(\forall x)(x \le x)$ (Integers)

4. **True or False** Very quickly, true or false. Draw Venn diagrams to convince yourself of your answers.

a) $\sim (\forall x)(P(x)) \equiv (\exists x)[\sim P(x)]$ Ans: T

b) $\sim (\exists x)(P(x)) \equiv (\forall x)[\sim P(x)]$ Ans: T

c) $\sim (\forall x)[\sim P(x)] \equiv (\exists x)[P(x)]$

d) $\sim (\exists x)[\sim P(x)] \equiv (\forall x)[P(x)]$

e) $(\forall x)[P(x) \Rightarrow Q(x)] \equiv \sim (\exists x)[P(x) \wedge \sim Q(x)]$

f) $\sim (\exists x)[P(x) \wedge Q(x)] \equiv (\forall x)[P(x) \Rightarrow \sim Q(x)]$

5. **Expanding Universes** In which of the universes $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are the following sentences true for *x* and *y* in those universes.

a) $(\forall x)(\exists y)(y = 1 - x)$

b) $(\forall x \ne 0)(\exists y)(y = 1/x)$

c) $(\exists x)(x^2 - 2 = 0)$

d) $(\exists x)(x^2 + 2 = 0)$

6. **Not as Easy as It Looks** Tell if the sentence

$$(\exists x \in U)[x \text{ is even } \Rightarrow 5 \le x \le 10]$$

is true or false in the following universe U.

e) $U = \{4\}$ Ans: F

f) $U=\{3\}$ Ans: T
g) $U=\{6,8,10\}$
h) $U=\{6,8,10,12\}$
i) $U=\{6,7,8,10,12\}$

7. **Small Universe** Which statements are true for the universe $U=\{1,2,3\}$.

a) $1<0 \Rightarrow (\exists x)(x<0)$ Ans: T
b) $(\exists x)(\forall y)(x \le y)$ Ans: F
c) $(\forall x)(\exists y)(x \le y)$
d) $(\exists x)(\exists y)(y = x+1)$
e) $(\forall x)(\forall y)(xy = yx)$
f) $(\forall x)(\exists y)(y \le x+1)$

8. **Well-Known Universe** Given the statements

$R(x)$: x is a rational number
$I(x)$: x is an irrational number

which of the following sentences are true in the universe of real numbers.

a) $(\forall x)[I(x) \vee R(x)]$ Ans: T
b) $(\forall x)[I(x) \wedge R(x)]$ Ans: F
c) $(\forall x)R(x) \vee (\forall x)I(x)$
d) $(\forall x)[R(x) \vee I(x)] \Rightarrow [(\forall x)R(x) \vee (\forall x)I(x)]$
e) $[(\forall x)R(x) \vee (\forall x)I(x)] \Rightarrow (\forall x)[R(x) \vee I(x)]$

9. **Famous Theorems** State the following famous theorems in the language of predicate logic.

a) **Intermediate Value Theorem** Let f be a continuous function on an interval $[a,b]$. If f changes sign from negative to positive on $[a,b]$, then there exists a number c between a and b such that $f(c)=0$.

b) **Fermat's Last Theorem** If n is an integer greater than 2, then there are no nonzero integer values of a,b,c that satisfy $a^n + b^n = c^n$.

c) **Euler's Theorem** If P is any regular polyhedron, and if v,e,f are the number of vertices, edges, and faces, respectively of the polyhedra, then $v-e+f=2$.

d) **Binomial Theorem** If a,b are real numbers and n an integer greater than or equal to zero, then

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}$$

10. **Negation** Negate the following sentences in words

a) All women are moral.
b) Every player on the team was over six feet tall.
c) For any real number y, there exists a real number x that satisfies $y = \tan x$.
d) There exists a real number x that satisfies $0 < x < 5$ and $x^3 - 8 = 0$.
e) The equation $a^n + b^n = c^n$ does not have nonzero integer solutions a, b, c for n a natural number $n > 2$.

11. **Negation in Predicate Logic** Negate the following sentences in symbolic form.

a) $(\forall x)[P(x) \Rightarrow Q(x)]$
b) $(\forall x)\left[x > 0 \Rightarrow (\exists y)(x^2 = y)\right]$
c) $P \Rightarrow (Q \wedge R)$
d) $(P \wedge Q) \Rightarrow R$

12. **Convergence and Non-convergence** Given the sentence:

A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges to L if and only if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|x_n - L| < \varepsilon)$$

a) State the negation of this sentence.
b) Using the negation found in a), prove that the sequence
c)

$$\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$$

does not converge to $L = 1/4$.

13. **Graph to the Rescue** If $P(x, y): y \le x^2 + 1$ where (x, y) are points in the plane, determine which of the following is true. Hint: A picture (i.e. graph) is worth a thousand words.

a) $(\forall x)(\forall y) P(x, y)$ Ans: F
b) $(\forall x)(\exists y) P(x, y)$ Ans: T
c) $(\exists x)(\forall y) P(x, y)$ Ans: F
d) $(\exists x)(\exists y) P(x, y)$

e) $(\exists y)(\forall x)P(x,y)$
f) $(\forall y)(\exists x)P(x,y)$

14. **Order Counts** Which of the following are true and which are false for real numbers x, y?

a) $(\forall x)(\forall y)(x < y)$
b) $(\exists x)(\forall y)(x < y)$
c) $(\forall y)(\exists x)(x < y)$
d) $(\exists x)(\exists y)(x < y)$

15. **Fun Time** State the denial of the words of wisdom attributed to Abraham Lincoln: "You can fool some of the people all the time and all the people some of the time, but you can't fool all the people all of the time."

16. **In Plain English** Restate the following sentences in plain English.

a) $(\forall x\in \mathbb{R})(\forall y\in \mathbb{R})\big[(x<y)\Rightarrow(\exists z\in \mathbb{R})\big[(x<z)\wedge(z<y)\big]\big]$
b) $(\forall x\in \mathbb{R})\big[(x>0)\Rightarrow(\exists y\in \mathbb{R})(x=y^2)\big]$
c) $(\forall m,n\in \mathbb{N})\big[(n>1)\Rightarrow\big[m\mid n\Rightarrow(m=1)\vee(m=n)\big]\big]$
d) $(\forall x\in \mathbb{R})(\forall y\in \mathbb{R})\big[(x<y)\Rightarrow(\exists z\in \mathbb{R})\big[(x<z)\wedge(z<y)\big]\big]$

17. **True Here, False There** Tell if the following sentences are true or false in each universe $\mathbb{N},\mathbb{Z},\mathbb{Q},\mathbb{R}$.

a) $(\forall x)(\exists y)(x<y)$
b) $(\forall y)(\exists x)(x<y)$
c) $(\exists x)(\forall y)(x<y)$
d) $(\exists y)(\forall x)(x<y)$
e) $(\forall x)\big[(x>0)\Rightarrow(\exists y)(y=x^2)\big]$

18. **Satisfiable in Predicate Logic** A **satisfiable** sentence is a sentence that is true in at least *some* universe. For example, the sentence $(\exists x\in U)(x>0)$ is satisfiable since it is true in the universe of real numbers. Tell if the following sentences are satisfiable, and if so give a universe.

a) $(\exists x\in U)\big[(x>0)\wedge(x<0)\big]$ Ans: NO
b) $(\exists x\in U)(x^2=-3)$ Ans: YES: $U=\mathbb{C}$
c) $(\exists x\in U)[P(x)\wedge \sim P(x)]$
d) $(\exists x\in U)\big[(x>0)\vee(x<0)\big]$

19. **Translation into Predicate Logic** Letting

$$E(x) = x \text{ is even}$$
$$O(x) = x \text{ is odd}$$

translate the following sentence to predicate logic.

a) Not every integer is even.
b) Some integers are odd.
c) Some integers are even and some integers are odd.
d) If an integer is even, then it is not odd.
e) If an integer is even, then the integer two larger is even.
f)

ΓΣΘΨΞΠΩ