# Section 1.6: Proof by Mathematical Induction

**Purpose of Section**: To introduce **the Principle of Mathematical Induction**, both the weak and the strong forms, and show how a certain class of theorems can be proven by this technique.

## Introduction

An important technique for verifying proofs in combinatorics and number theory is the Principle of Mathematical Induction. The technique was used implicitly in Euclid's Elements in a "descent proof" that every natural number has a prime divisor. The term Mathematical Induction was first coined in 1828 by the English logician Augustus DeMorgan (1806-1871) in an article called *Induction.*

Mathematical Induction is generally not used in deriving new formulas, but is an effective tool to verify formulas and facts you suspect are true. That said, it part of the repertoire of any good mathematician.

The beauty of mathematical induction is it allows a theorem to be proven in cases when there are an infinite number of cases to explore without having to examine each case. Induction is the mathematical situation analogous to an infinite row of dominoes where if you tip over the first one and if each domino tips over its adjacent one, they all get tipped over. The nice thing about induction is you don't have to prove it works, it's an axiom[1] in the foundations of mathematics.

Mathematical induction provides a convenient way to establish that a statement is true for all natural numbers 1,2,3,…The following statements are prime candidates for proof by mathematical induction.

- ▶ For all natural numbers $n$, $1+3+5+\cdots+(2n-1)=n^2$
- ▶ If a set A contains $n$ elements, then the collection of subsets of A contains $2^n$ elements.
- ▶ $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\cdots\cdot\frac{2n-1}{2n}\le\frac{1}{\sqrt{3n+1}}\cdot$ for all natural numbers $n$.

[1] In 1889 Italian mathematician Guiseppe Peano (1858-1932) published a list of five axioms which define the natural numbers. Peano's $5^{th}$ axiom is called the induction axiom, which states that "*any property which belongs to* 1 *and also to the successor of any number which has the property belongs to all numbers*."

Here, then is how the method of mathematical induction works.

**Mathematical Induction**

The Principle of Mathematical Induction is a method of proof for verifying that a proposition $P(n)$ is true for all natural numbers $n = 1, 2, \ldots$. The methodology for proving theorems by induction is as follows.

**Methodology of Mathematical Induction**

To verify that a proposition $P(n)$ holds for all natural numbers $n$, the **Principle of Mathematical Induction** consists of carrying out two steps.

- **Base Case:** Prove $P(1)$ is true.
- **Induction Step:** Assume $P(n)$ is true for an arbitrary $n$, then prove $P(n+1)$ is true.

If the above two steps are proven, then by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers $n$. In other words:

$$\left.\begin{array}{c} P(1) \text{ is true} \\ (\forall n \in \mathbb{N})\left[P(n) \text{ true} \Rightarrow P(n+1) \text{ true}\right] \end{array}\right\} \Rightarrow (\forall n \in \mathbb{N})\, P(n) \text{ is true}$$

**Important Note:** Don't confuse *mathematical induction* with *inductive reasoning* associated with the natural sciences. Inductive reasoning in the sciences is a scientific method whereby one *induces* general principles from specific observations. Mathematical induction is not the same thing: it is a *deductive* form of reasoning used to establish the validity of a proposition for all natural numbers.

**Important Note:** There are many modifications of the basic induction proof. For example, there is no reason the base case starts with $P(1)$. If the base case is replaced by $P(a)$, where "$a$" is any integer (positive or negative), one would conclude $P(n)$ true for all $n \geq a$. Also, if the induction step is replaced by the implication $P(n) \Rightarrow P(n+2)$, one concludes $P(n)$ true for $P(1), P(3), \ldots, P(2n+1), \ldots$

**Theorem 1: Famous Identity** If $n$ is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

**Proof:** Denote $P(n)$ as

$$P(n): 1+2+\cdots+n=\frac{n(n+1)}{2}$$

**Base Case:** $P(1)$ is true since[2] $P(1)$ says

$$1=\frac{1\cdot(2)}{2}$$

**Induction Step:** Assume $P(n)$ true for an arbitrary $n$:

$$P(n):\ 1+2+\cdots+n=\frac{n(n+1)}{2}$$

Adding $n+1$ to each side of this equation, we get:

$$\begin{aligned}1+2+\cdots+n+(n+1)&=\frac{n(n+1)}{2}+(n+1)\\&=\frac{n(n+1)+2(n+1)}{2}\\&=\frac{(n+1)(n+2)}{2}\end{aligned}$$

which is statement $P(n+1)$. Hence $P(n)\Rightarrow P(n+1)$ and so by induction the result is proven. ▌

**Visual Proof** The $n\times n$ array[3] drawn in Figure 1 has $n^2$ boxes where

- number of boxes with $x's$ = is $1+2+3+\cdots+n$
- number of unmarked boxes = $1+2+\cdots+(n-1)$,

Hence,

$$n^2=(1+2+\cdots+n)+(1+2+\cdots(n-1))$$

Adding $n$ to each side of this equation gives

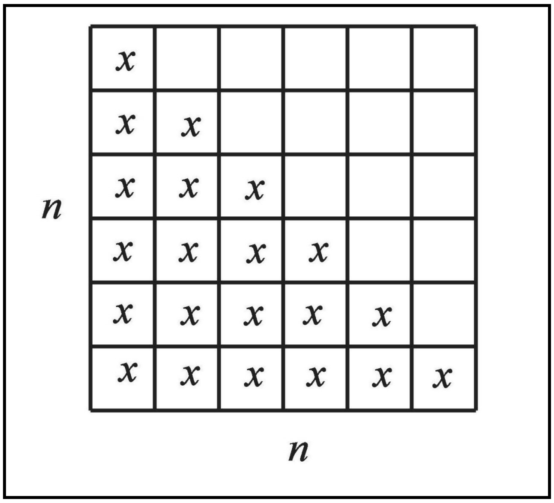
$$n^2+n=2(1+2+\cdots+n)$$

and solving for $1+2+3+\cdots+n$ gives the desired result

[2] The reader can verify that P(2) and P(3) are also true, but that isn't relevant to proof by induction.

[3] The array is really a $6\times 6$ array but we imagine it is an $n\times n$ array.

$$1+2+3+\cdots+n=\frac{n^2+n}{2}=\frac{n(n+1)}{2}$$




Visual proof
Figure 1

> **Important Note:** Do we have to prove that the principle of mathematical induction holds? The answer is no. We accept mathematical induction as a logical axiom in much the same way as we accept the classical rules of Aristotelian logic.

The following theorem is one where a direct proof would be difficult but induction is easy.

**Theorem 2: Induction in Calculus** Prove that for every natural number $n$, we have

$$P(n): \quad \frac{d^n\left(xe^x\right)}{dx^n}=\left(x+n\right)e^x$$

**Proof:** Using mathematical induction,we have

**Base Step**: If $n=1$ and using the product rule for differentiation, we can write

$$\frac{d\left(xe^x\right)}{dx}=x\frac{d}{dx}e^x+e^x=\left(x+1\right)e^x \ .$$

**Induction Step:** Assuming

$$P(n): \frac{d^n\left(xe^x\right)}{dx^n} = \left(x+n\right)e^x$$

true for arbitrary $n$, we compute

$$\begin{aligned} P(n+1): \quad \frac{d^{n+1}\left(xe^x\right)}{dx^{n+1}} &= \frac{d}{dx}\left(\frac{d^n\left(xe^x\right)}{dx^n}\right) \\ &= \frac{d}{dx}\left[\left(x+n\right)e^x\right] \quad \text{(induction assumption)} \\ &= \left(x+n\right)e^x + e^x \quad \text{(product rule)} \\ &= \left[x+\left(n+1\right)\right]e^x \end{aligned}$$

which proves $P(n+1)$, hence by induction the theorem is proven ▌

**Theorem 3: Inequality by Induction** If $n \geq 5$, then $2^n > n^2$.

**Proof:** Defining $P(n): 2^n > n^2$ we prove:

**Base Case:** $P(5): 2^5 = 32 > 25 = 5^2$.

**Induction Step:** $P\left(n\right) \Rightarrow P\left(n+1\right)$ for $n \geq 5$.

This step requires we prove $2^n > n^2 \Rightarrow 2^{n+1} = \left(n+1\right)^2$, $n \geq 5$ which we do with the following steps:

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &> 2n^2 \quad \text{(induction hypothesis)} \\ &= n^2 + n^2 \\ &\geq n^2 + 5n \quad \text{(we assume } n \geq 5\text{)} \\ &= n^2 + 2n + 3n \\ &> n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

By induction $P(n)$ true for all $n \geq 5$. ▌

So how did we arrive at the non-intuitive inequalities in the previous proof that made everything turn out so nice? The answer is we worked out the inequalities backwards starting at the conclusion.

Sometimes a result can be proven by induction or with a direct proof. The following problem is such an example. You can decide if you have a preference.

**Theorem 4: Direct Proof or Proof by Induction?** For any $n \in \mathbb{N}$, the number $n(n+1)$ is even.

**Direct Proof**: The idea is to show that for any natural number $n$ the number $n(n+1)$ contains a factor 2

If $n$ is even, we have

$$(\exists k \in \mathbb{N})(n = 2k) \Rightarrow n(n+1) = 2k(n+1)$$

If $n$ is odd, we have

$$(\exists k \in \mathbb{N})(n = 2k+1) \Rightarrow n(n+1) = (2k+1)(2k+2) = 2(k+1)(k+2)$$ ▌

**Proof by Induction:** Let $P(n): n(n+1)$ is even . We show

**Base Step:** $P(1): 1(1+1) = 2$ is even
**Induction Step:** $P(n) \Rightarrow P(n+1)$ Assuming $n(n+1)$ is even, we have $n(n+1) = 2k$ for $k \in \mathbb{N}$ .Hence, we can write

$$\begin{aligned} P(n+1):\ (n+1)(n+2) &= n(n+1) + 2(n+1) \\ &= 2k + 2(n+1) \\ &= 2(k+n+1) \end{aligned}$$

which proves, $P(n+1)$, and so by induction the result is proven. ▌

| **Important Note:** Someone once said mathematical induction is the formal way of saying "and so on."? |
|---|

The type of induction discussed thus far is sometimes called **weak induction**. We now introduce another version of induction called **strong induction**. Although the two versions are logically equivalent, there are problems where strong induction is more convenient.

**Strong Induction**

The difference between weak and strong induction has to do with what is assumed in the induction step. In weak induction – using the dominoes metaphor – if any domino is tipped over, then the next one is tipped over. In strong induction, you assume *all previous dominoes* are tipped over, then prove the next one is tipped over. Surprising as it might seem, both weak and strong induction are logically equivalent, the difference is more practical, sometimes strong induction is more convenient and sometimes weak induction is more convenient. The following examples illustrate why strong induction is the desired form of induction in some proofs.

**Methodology of (Strong) Mathematical Induction**

To verify a proposition $P(n)$ holds for all natural numbers $n$, the **Principle of (Strong) Mathematical Induction** consists of carrying out the following steps.

1. **Base Case:** Prove that $P(1)$ is true.
2. **Induction Step**: Show that for all $n \in \mathbb{N}$

$$P(1) \wedge P(2) \wedge \cdots \wedge P(n) \Rightarrow P(n+1).$$

**Theorem 5:** Every integer greater than 1 is divisible by a prime number.

**Proof:**

**Base Case**: The result is true for $n = 2$ since 2 is prime and 2 divides 2..
**Induction Step**: Assume all positive integers from 2 through $n-1$ are divisible by a prime number, where $n$ is an arbitrary natural number. The goal is to show $n$ is divisible by a prime number. If $n$ is prime, then it is divisible by a prime number, itself. If $n$ is not prime, then it is a composite and has a divisor $m$ which is not 1 or $n$. By the induction assumption, we know there is a prime number $p$ that divides $m$. But $m$ divides $n$, so $p$ divides $n$. Symbolically,

$$[(p \mid m) \wedge (m \mid n] \Rightarrow (p \mid n) \ .$$

Hence, the induction step in proven, so by the Principle of Strong Induction, the result is proven. ▌

A fundamental result in number theory is the **Fundamental Theorem of Arithmetic**, which can be proven by strong induction.

**Theorem 6: Fundamental Theorem of Arithmetic**

Every natural number $n \geq 2$ can be written as the product of prime numbers. For example, $350 = 2 \times 5^2 \times 7$, $1911 = 3 \times 7^2 \times 13$.

**Proof:** We must prove

**Base Case:** $P(2)$ holds since 2 is prime.

**Induction Step:** For an arbitrary natural number $n$ we prove

$$P(2) \wedge P(3) \wedge P(4) \wedge \cdots \wedge P(n) \Rightarrow P(n+1)$$

We assume $P(2), P(3), ..., P(n)$ true which means every natural number 2,3,… $n$ can be written as the product of primes. To prove $n+1$ can be written as a product of prime numbers, consider two cases.:

**Case 1:** If $n+1$ is a prime number, the result is proven since we can write $n+1 = n+1$ .

**Case 2:** If $n+1$ is not prime, it can be written as the product $n+1 = pq$, where both factors $p$ and $q$ are less than $n+1$ and greater and or equal to 2. By the induction hypothesis both $p$ and $q$ can be written as the product of primes:

$$p = p_1 p_2 \cdots p_m \qquad q = q_1 q_2 \cdots q_n$$

Hence, we have

$$n+1 = pq = (p_1 p_1 \cdots p_m)(q_1 q_2 \cdots q_n) \ .$$

which proves $P(n+1)$ true, so by the principle of strong induction $P(n)$ is true for all $n \geq 2$ . ▌

**History of Mathematical Induction** Although some elements of mathematical induction have been hinted at since the time of Euclid, one of the oldest argument using induction goes back to the Italian mathematician *Francesco Maurolico*, who used induction in 1575 to prove that the sum of the first $n$ odd natural numbers is $n^2$. The method was later discovered independently by the Swiss mathematician *John Bernoulli*, and French mathematicians *Blaise Pascal* (1623-1662) and *Pierre de Fermat* (1601-1665). Finally, in 1889 the Italian logician Guiseppe Peano (1858-1932) laid out five axioms for deducing the natural numbers, of which his fifth axiom was the Principle of Mathematical Induction. Hence, from that point of view induction is an axiom of arithmetic.

The next example shows a variation of the base step from previous examples. Each problem is different and you must adjust the induction proof accordingly.

**Theorem 7: Modifying the Base Step** You are given two rulers, one is 3 units long, the other is 5 inches long. Your job is to show you can measure any unit distance greater than or equal to 8 using only those rulers.

**Proof:** Let $P(n)$ be the proposition

$P(n) =$ any integer distance of length $n \geq 8$ can be measured with rulers, lengths 3,5 It is useful to see the following pattern that develops.

$$P(8) = 5+3$$
$$P(9) = 3+3+3$$
$$P(10) = 5+5$$
$$P(11) = P(8)+3 = (5+3)+3$$
$$P(12) = P(9)+3 = (3+3+3)+3$$
$$P(13) = P(10)+3 = (5+5)+3$$

This pattern will serve as an aid in deciding the base and induction steps which is often the most difficult part in an induction proof.

**Base Step:** For the base step, we verify the first *three* propositions:

$$P(8) = 5+3, \quad P(9) = 3+3+3, \quad P(10) = 5+5$$

**Induction Step:** We now prove the induction step

$$P(8), P(9), \ldots, P(n) \Rightarrow P(n+1), \ \ n \geq 10$$

To prove this step, we make the observation that if $P(n-2)$ is true (i.e. a length of $n-2$ can be measured with rulers of length 3 and 5), then $P(n+1)$ is also true since a length of $n+1$ is 3 units longer than $n-2$. Hence, $P(11)$ is true since $P(8)$ is true, and $P(12)$ is true since $P(9)$ is true, and so on. Hence, if $P(8), P(9), \ldots, P(n)$, $n \geq 10$ is true so is $P(n+1)$. Hence, by induction $P(n)$ is true for all natural numbers $n$.

## Problems

1. **Proof by Induction** Prove the following propositions, either by weak or strong induction.

a) $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

b) $P(n): 1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$

c) $1+3+5+\cdots+(2n-1)=n^2$ .

d) $9^n-1$ is divisible by 8 for all natural numbers $n$.

e) For $n\geq 1,\ 1+2^2+2^3+2^4+\cdots+2^n=2^{n+1}-1$

f) For $n\geq 5,\ 4n<2^n$,

g) $n^3-n$ is divisible by 3 for $n\geq 1$.

h) $2^{n-1}\leq n!,\ n\in\mathbb{N}$

i) For all positive integers $n$, $n^2+n$ is even.

j) For any real numbers $a,b$ and natural number $n$, we have

$$(ab)^n=a^nb^n.$$

2. **Something Fishy** Let's prove by induction $n^2+7n+3$ is even for all natural numbers $n=1,2,...$ .What is wrong with the following induction argument? Letting $P(n)$ denote

$$P(n): n^2+7n+3 \text{ is an even integer}$$

we prove $P(n)\Rightarrow P(n+1)$. Assuming $P(n)$ true, we have

$$\begin{aligned} P(n+1):\ (n+1)^2+7(n+1)+3 &= (n^2+2n+1)+7n+7+3 \\ &= (n^2+2n+3)+2(n+4) \\ &= 2k+2(n+4) \quad (\text{since } P(n) \text{ even}) \\ &= 2(k+n+4) \end{aligned}$$

Hence $P(n+1)$ is true which by induction proves $n^2+7n+3$ is even for all natural numbers $n$.

3. **Clever Mary** To prove the identity

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Mary evaluates the left-hand side of the equation for $n=0,1,2$ getting

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $p(n)$ | 0 | 1 | 3 |

.

and then finds the quadratic polynomial $p(n)=an^2+bn+c$ that passes through those points, getting

$$p(n)=\frac{1}{2}n^2+\frac{1}{2}n=\frac{n(n+1)}{2}.$$

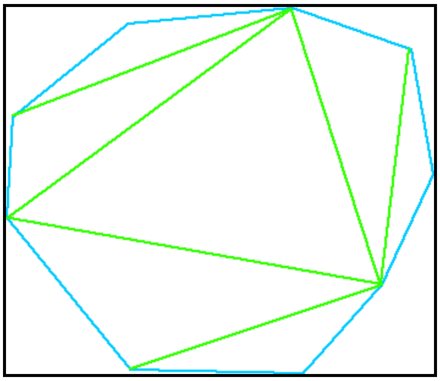
Mary turned this into her professor. Is her proof[4] valid?

4. **Hmmmmmmmm** Is there something fishy with this argument that Mary can carry a 50-ton load of straw on her back. Clearly she can carry one straw on her back, and if she can carry $n$ straws on her back, she can certainly carry one more. Hence, she can carry any number of straws on her back which can amount to a 50-ton load.

5. **Geometric Principle by Induction** Show that every convex polygon[5] can be divided into triangles. An example illustrating a triangulation (triangulations are not unique) of a 8-sided convex polygon is drawn below..

[4] This problem is based on a problem in the book $A = B$ by Marko Petkovsek, Doron Zeilberger and Herbert Wilf. (This amazing book, incidentally, can be downloaded free on the internet.)

[5] A convex polygon is a simple polygon (sides do not cross) whose interior is a convex set.(i.e the line segment connecting any two points in the set also belongs to the set.



6. **Nature of Induction** Often one gets a general idea something is true by constructing examples. For example, suppose you have only 3 and 5 cents stamps and want to determine what postages are possible. So you make a table like

| | 5 CENT STAMP | | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | 4 | 5 |
| 3 CENT STAMP | 0 | 0 | 5 | 10 | 15 | 20 | … |
| | 1 | 3 | 8 | 13 | 18 | 23 | … |
| | 2 | 6 | 11 | 16 | 21 | 26 | … |
| | 3 | 9 | 14 | 19 | 24 | 29 | … |
| | 4 | 12 | 17 | 22 | 27 | 32 | … |
| | 5 | … | … | … | … | … | … |

From this table, we might hypothesize that possible postages are 0,3,5, and 6 cents and every value of 8 or more cents. Can you prove this by induction?

**Answer:** If we denote

$$P(n): \text{ postage of } n \text{ cents is possible}$$

then the goal is to prove $P(n)$ for $n = 0, 1, 2, \ldots$ We can verify the initial step and show $P(n)$ is true for $n = 0, 3, 5, \text{ and } 6$ and false for $n = 1, 2, 4, 5, \text{ and } 7$. We now resort to strong induction and *assume* all postages are possible for $8, 9, \ldots, n$ cents and prove a postage worth $n+1$ cents is possible. To do this, consider four cases:

| |
|---|
| case 1: $n+1=8$ (true, we can use a 3 cent and a 5 cent stamp) |
| case 2: $n+1=9$ (true, we can use three 3 cent stamps) |
| case 3: $n+1=10$ (true, we can use two 5 cent stamps) |
| case 4: $n+1>10$ (same as $n-2\geq 8$ which is assumed true) |

So we have proven $P(0)\wedge P(1)\wedge P(2)\wedge\cdots P(n)\Rightarrow P(n+1)$ and so by strong induction $P(n)$ is true for all $n=0,1,2,...$ Note: The basic idea is that for $n>10$ (or $n=11,12,13,...$), we know postage of $n-2$ = $8,9,10,...$ (three cents less) is possible, so we simply add another 3 cent stamp to get a postage of $n+1$.

7. **Fibonacci Sequence** The Fibonacci sequence

$$\{F_n, n=1,2,...\}$$

is defined for $n\geq 2$ by the equations

$$F_{n+1}=F_n+F_{n-1},\ F_1=1, F_2=1.$$

A few terms of the sequence are 1,1,2,3,5,8,13, Show the $n$th term of the sequence is given by

$$F_n=\frac{\alpha^n-\beta^n}{\sqrt{5}}$$

where $\alpha=\left(1+\sqrt{2}\right)/2,\ \beta=\left(1-\sqrt{2}\right)/2$.

> **Parting Note:** Just because something is true for the first million numbers doesn't mean it's true for the millionth and one number. For example, the equation
>
> $$(n-1)(n-2)\cdots(n-1{,}000{,}000)=0$$
>
> is true for $n$ from 1 to a million, but not true when $n=1{,}000{,}001$.

8. **Peano's Axioms** The Principle of Mathematical Induction is generally taken as an axiom for the natural numbers. and is in fact the fifth axiom for **Peano's axioms**. Google Peano's axioms and read about them on the internet.

ΓΣΘΨΞΠΩ