Section 2.1 Basic Operations of Sets

Purpose of Section We present an informal discussion of some of the basic ideas related to sets, including membership in sets, subsets, the empty set, power set, union and intersection, and other fundamental concepts. The material presented in this section will provide a background for study in all areas of mathematics.

Sets and Membership

Sets are (arguably) the most basic of all mathematical objects. Quite simply, a set is

a collection things, the things belonging to the set are referred to as members

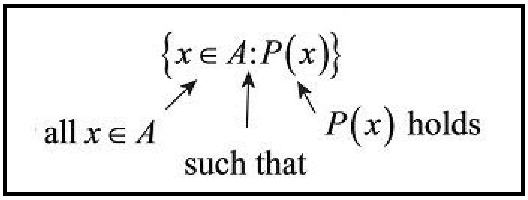
or elements of the set. Other synonyms for the word "set," are *collection*, *class, family*, and *ensemble*. Hence, we refer to a collection of people, family of functions, an ensemble of voters, and so on. We even consider sets whose members themselves are sets, such as the set of all classes at a university, where each class consists of students. In mathematics, we might consider the set of all open intervals on the real line.

If a set does not contain too many members, we can specify the set by simply writing down the members inside a pair of brackets, such as $\{3, 7, 31, 127, 8191\}$, which denotes the first five Mersenne primes[1]. Sometimes sets contain an infinite number of elements, like the natural numbers, where we might specify them by $\{ 1,2,3,\cdots\}$, where the three dots after the 3 signify "and so on" which denotes the fact that the sequence is never ending.

We generally denote sets by capital letters such as $A,B,C,...$ and members of sets by small letters, like $a,b,c,...$. If a is a member of a set A, we denote this fact by writing $a \in A$ and we say "a is a member of A" or "a belongs to A". If an element does not belong to a set, we denote this by $a \notin A$.

One can also specify a set by specifying defining properties of the member of the set, such as

[1] Mersenne primes are prime numbers of the form $2^n - 1,\ n = 1,2,\ldots$. At the present time 2009, there are 47 known Mersenne primes., the largest being $2^{43,112,609} - 1$ which has 12,978,189 digits. It is unknown whether there are an infinite number of Mersenne primes.



which we read as “the set of all x in a set A that satisfies the condition $P(x)$.” The set of even integers could be denoted by

$$\text{Even integers} = \{n \in \mathbb{Z} : n \text{ is an even integer}\}.$$

Some common sets in mathematics are the following.

Common Sets in Mathematics
$\mathbb{N} = \{1, 2, 3, \ldots\}$ (the natural numbers or positive integers)
$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (integers)
$\mathbb{Q} = \{n : n = p/q,\ p \text{ and } q \neq 0 \text{ integers}\}$ (rational numbers)
$\mathbb{R} =$ the set of real numbers
$\mathbb{C} =$ the set of complex numbers
$(a,b) = \{x \in \mathbb{R} : a < x < b\}$ (open interval from a to b)
$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed interval from a to b)
$(a,\infty) = \{x \in \mathbb{R} : x > a\}$ and $(-\infty,a) = \{x \in \mathbb{R} : x < a\}$ (open rays)
$[a,\infty) = \{x \in \mathbb{R} : x \geq a\}$ and $(-\infty,a] = \{x \in \mathbb{R} : x \leq a\}$ (closed rays)

The half-open intervals $(a,b]$ and $[a,b)$ are defined similarly.

Some examples illustrating membership and non-membership in sets are the following.

$$\pi \in \mathbb{R}$$
$$\pi \notin \mathbb{Q}$$
$$\frac{7}{12} \in \mathbb{Q}$$
$$e \in \{x \in \mathbb{R} : x \text{ is transcendental}\}$$
$$0 \notin \{x \in \mathbb{R} : 0 < x < 1\}$$

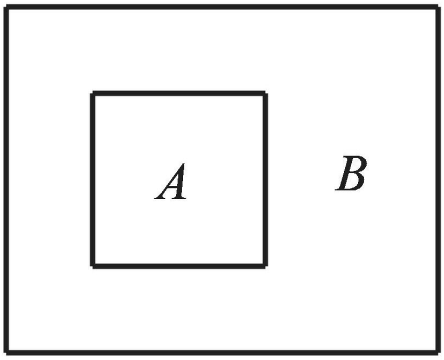
Rigor in Mathematics In the study of mathematics, there should always be a sufficient degree of precision or rigor, which means mathematical concepts should be logically precise. However, it is also important to built up intuition about mathematical concepts. The great Swiss mathematician Leonhard Euler had an uncanny intuition about concepts and often did not prove results he thought to be true. That said, no mathematician would ever say he wasn't one of the greatest mathematicians who ever lived.

Universe, Subset, Equality, Complement, Empty Set

▶ **Universe:** The universe U is the set consisting of the **totality of elements** under consideration. A common universe in number theory would be the natural numbers $\mathbb{N}$, whereas in calculus a common universe would be the real numbers $\mathbb{R}$, or maybe an interval on the real line like $[0,1]$ or $[0,\infty)$.

▶ **Subset:** One is often interested in a set A which is part of a larger set B. We say that a set A is a **subset** of a set B if every member of A is also a member of B. Symbolically, we write this as $A \subseteq B$ and is read "A is contained in B." If $A \subseteq B$ and $A \neq B$ we say that A is a **proper subset** of a set B and denote this by $A \subset B$.

Sets are often illustrated visually by **Venn diagrams**, where sets are represented by squares or circles and elements of a set as points inside the square or circle. Figure 1 shows a Venn diagram which illustrates $A \subseteq B$.



Venn Diagram Illustrating $A \subseteq B$
Figure 1

Important Note: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

▶ **Equality of Sets:** Two sets A and B are **equal** if they consist of the same members. In other words,

$$A = B \Leftrightarrow (x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A).$$

Stated another way: $A = B \iff A \subseteq B$ and $B \subseteq A$.

▶ **Complement of a Set:** If a set A belongs to a universe U, the **complement** of A, written $\overline{A}$ consists of all members of U that do not belong to A. That is,

$$\overline{A} = \{x \in U : x \notin A\}$$

▶ **Empty Set**: The set that doesn't contain any members is called the **empty set**[2] (or **null set**), and plays a special roll in set theory. It is denoted by the Greek letter $\varnothing$ or by the empty bracket $\{\ \}$.

Note that we say "the" empty set rather than "an" empty set. The reason for this language is because two sets are equal if they contain the same members, and since the empty set contains no members, there is only *one* empty set. Just remember, the empty set is *not nothing*, it is *something*, it's just that it contains *nothing*. You might think of the empty set as a bag that has nothing in it. In this regard, it is best to denote the empty set by $\{\ \}$ rather than $\varnothing$. With this interpretation, we can write

$$\{\text{all people over 500 years old}\} = \{x \in \mathbb{R} : x^2 + 1 = 0\}$$

Seems strange but it is logically correct since both sets are the empty set..

Important Note: $\in$ versus $\subseteq$: When you ask if $a \in A$ you ask the question is "a" a *member* of A; when you ask if $B \subseteq A$ you are asking something different, you are asking if every member *of* B is also a member of A. For example,

$$\{a,\{a\}\} \notin \{a,\{a\}\} \text{ but } \{a,\{a\}\} \subseteq \{a,\{a\}\}.$$

▶ **Power Set**: For every set A, the collection of *all* subsets of A is called the **power set** of A and denoted by $P(A)$ or 2^A. For example, the power set of the set $A = \{a,b,c\}$ consists of the set of eight subsets of A:

$$P(A) = \{\varnothing,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}.$$

A few power sets of some other sets are given in Table 1

2 2 If you object to a set with no elements, you are like the persons in the past who objected to the number 0, since it stood for nothing. The number 0 was resisted for centuries as a legitimate number.

Set	Power Set
$\varnothing$	$\{\varnothing\}$
$\{a\}$	$\{\varnothing,\{a\}\}$
$\{a,b\}$	$\{\varnothing, \{a\},\{b\},\{a,b\}\}$
$\{a,\{b\}\}$	$\{\varnothing, \{a\},\{\{b\}\},\{a,\{b\}\}\}$

Power Sets
Table 1

Later, we will prove that for a set of n elements, the power set contains 2^n elements, whose proof looks like a candidate for induction[3].

> **Important Note:** Often the power set of a set A is denoted by 2^A since sets that contain n elements have 2^n subsets.

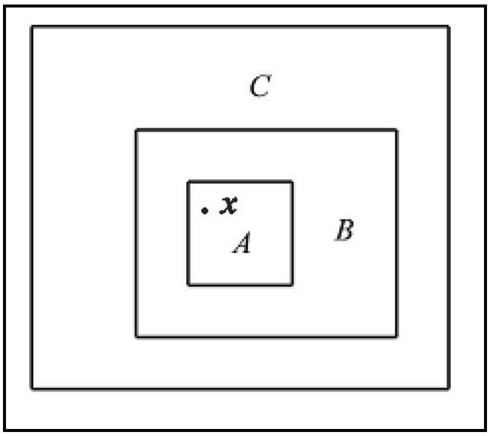
Theorem 1: Guaranteed Subset Every set contains at least one subset since for any set A, we have $\varnothing \subseteq A$.

Proof We must show $x \in \varnothing \Rightarrow x \in A$. Our job is finished before we start since the hypothesis $x \in \varnothing$ is false, hence the implication is true.

Theorem 2: Transitive Subsets For sets A, B and C, the following property holds:

$$\left[(A \subseteq B) \wedge (B \subseteq C)\right] \Rightarrow A \subseteq C.$$

Proof: Let $x \in A$. Since $A \subseteq B$, we know $x \in B$. But $B \subseteq C$ and so $x \in C$. Hence $A \subseteq C$. Figure 2 illustrates this result with a Venn diagram.



[3] Things get a lot more interesting when we consider the family of all subsets of an infinite set like the natural numbers. It turns out that … well, we don't want to ruin the fun for you.

Venn diagram illustrating $A \subseteq B \subseteq C$
Figure 2

Important Note: One reason the concept of a set is so powerful is the fact that the elements can be anything, even sets themselves. In the area of mathematics called *analysis*, sets normally consist of sets of numbers, like the integers, real numbers, intervals on the real line, and so on. In geometry they are geometric objects, in probability they are sample spaces and events, and so on. In topology, one studies certain *families of subsets* of a given set, called the *open sets* of the set.

Note: $5 \neq \{5\}$, $\{a,b\} \notin \{a,b\}$, $\varnothing \notin \varnothing$, $\varnothing \neq \{\varnothing\}$

Example 1: Membership and Subset Do you understand why the following are correct?

$a)$ $\varnothing \subseteq \{x \in \mathbb{R} : x^2 = -1\}$

$b)$ $3 \in \mathbb{N}$

$c)$ $-1 \notin \mathbb{N}$

$d)$ $\pi \notin \mathbb{Q}$

$e)$ $e \in \mathbb{R}$

$f)$ $3.5 \in \mathbb{C}$

$g)$ $3 + 2i \notin \mathbb{R}$

$h)$ $\{x : x^2 - 1 = 0\} \not\subset \{x : x^3 - 1 = 0\}$

Example 2: Membership vs Subsets In the following examples, which are members of a set and which are subsets of a set.

a) $\varnothing \in \{\varnothing, \{\varnothing\}\}$ Answer : Yes

b) $\varnothing \subseteq \{\varnothing, \{\varnothing\}\}$ Answer: Yes

c) $\varnothing \in \{\{\varnothing\}\}$ Answer: No

d) $a \in \{\{a\}, \{a, \{a\}\}\}$ Answer: No

e) $\{a\} \in \{\{a\}, \{a, \{b\}\}\}$ Answer: Yes

f) $\{\varnothing\} \subseteq \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ Answer: No

g) $\{a, \{b\}\} \in \{a, \{a, \{b\}\}\}$ Answer: Yes

h) $\{a,\{b\}\}\in\{\{b\},a\}$ Answer: No

Theorem 3: Power Set For any two sets A and B, we have

$$A\subseteq B \Leftrightarrow P(A)\subseteq P(B) \ .$$

Proof:

$(A\subseteq B)\Rightarrow(P(A)\subseteq P(B))$

We start by taking an arbitrary set $X\in P(A)$. Since $X\in P(A)$ then by definition $X\subseteq A$ and by hypothesis $X\subseteq B$. But this means $X\in P(B)$ and $P(A)\subseteq P(B)$.

$(A\subseteq B)\Leftarrow(P(A)\subseteq P(B))$

If $x\in A$, then $\{x\}\in P(A)$, and by assumption $P(A)\subseteq P(B)$, we have $\{x\}\in P(B)$, which means $x\in B$ and hence $A\subseteq B$. ∎

Union, Intersection, and Difference of Sets

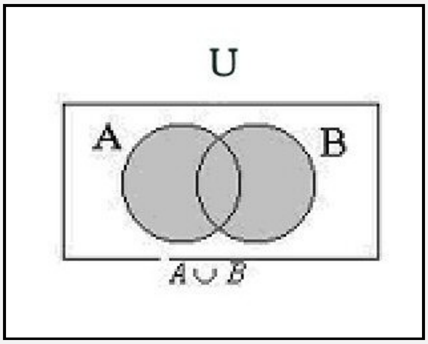
In traditional arithmetic and algebra, we use the binary operations of $+$ and $\times$. In logic, we have the analogous binary operations of $\vee$ and $\wedge$. For sets, we have the binary operations of union $\cup$ and intersection $\cap$

Definition

Let A,B be subsets of some universe U

- **Union:** The **union** of two sets A and B, denoted $A\cup B$, is the set of elements that belong to *A or B* or both.[4]

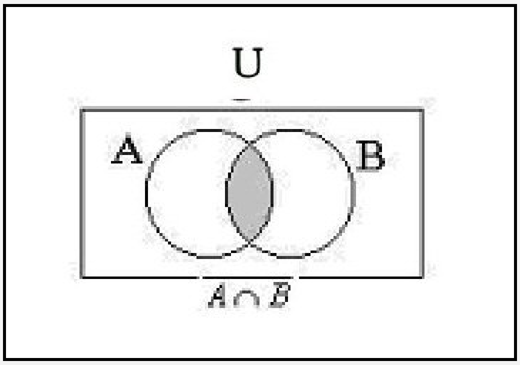
$$A\cup B=\{x\in U : x\in A \text{ or } x\in B\}$$



- **Intersection:** The **intersection** of two sets A and B, denoted $A\cap B$, is the set of elements that belong to *A and B.*

[4] This "or" is the inclusive "or" in contrast to the exclusive "or" which means one or the other but not both.

$$A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$$



If $A \cap B = \varnothing$ the sets A and B are called **disjoint**.

- The **difference** of two sets, denoted $A - B$, is defined to be the set of elements that belong to A but not B.

$$A - B = \{x \in U : x \in A \text{ and } x \notin B\}$$

Important Note: The Italian mathematician Giuseppe Peano introduced the notation for set inclusion $(\in)$, set union $(\cup)$ and set intersection $(\cap)$ in 1889 in a treatise on axioms for the natural numbers.

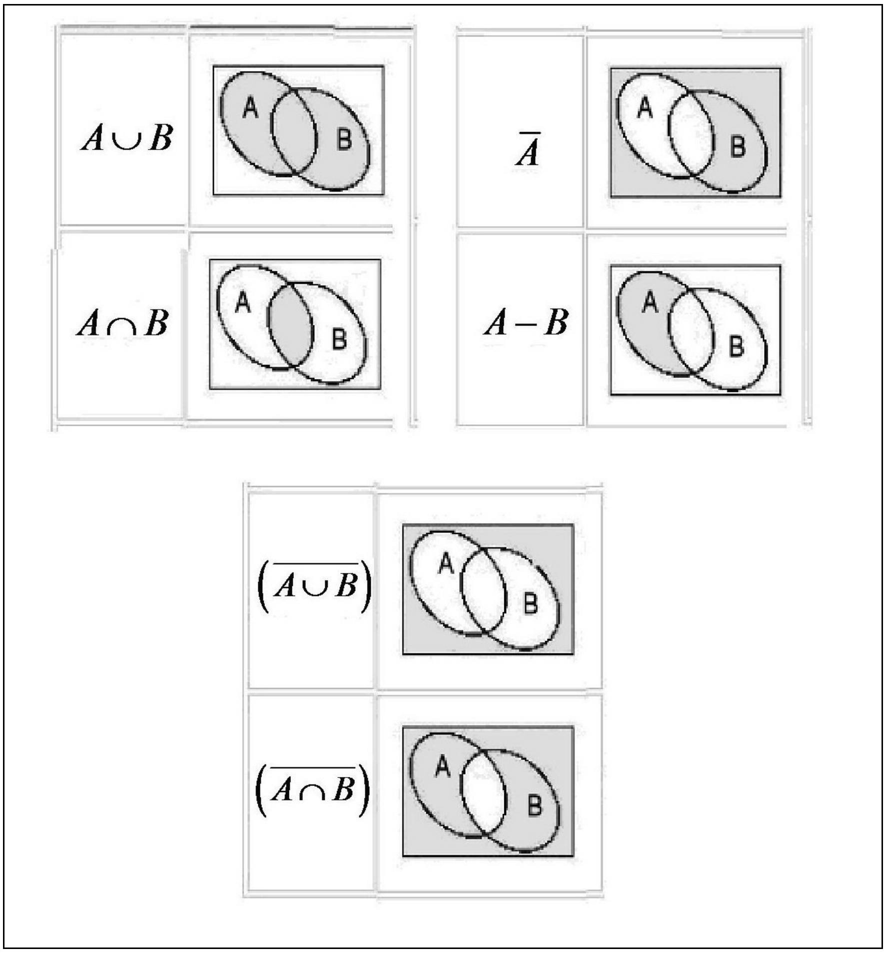
Historical Note: Although the origin of the idea of set is vague, Greek mathematicians defined a circle as

"A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another."

When the English mathematician George Boole referred to sets in 1854 in his seminal treatise, *An Investigation of the Laws of Thought*, the concept of a "set" *per se* was well established. That said, the distinction between "finite sets" and "infinite sets" eluded mathematicians over the centuries until the late 1800s in the work of the German mathematician George Cantor, whose words we study in Sections 2.4 and 2.5.

Venn Diagrams of Various Sets

Venn diagrams[5] for subsets A, B of a universe U are shown below in Figure 3. The universe is represented by the rectangle containing the sets.

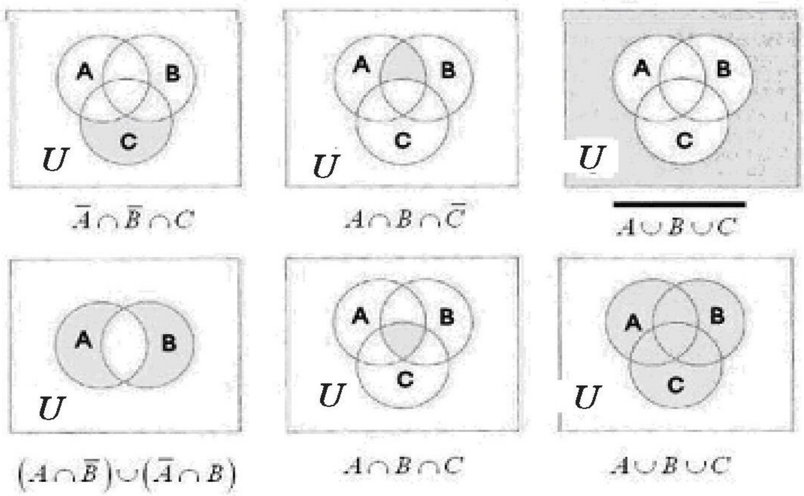


Venn diagrams for two sets
Figure 3

Historical Note: George Venn (1834-1923) was an English mathematician who further developed George Boole's symbolic logic but is known mostly for his pictorial representations of the relations between sets.

[5] Venn diagrams were the invention of British logician John Venn (1834-1923) who made major contributions to logic and probability. John Venn was an ordained minister but gave up the ministry in 1883 to concentrate on mathematics and logic.

Figure 4 illustrates Venn diagrams for three subsets of a universe .



. .

Venn diagrams for three sets
Figure 4

Naive versus Axiomatic Set Theory
Naive set theory, as we introduce in this section, studies basic properties of sets, such as complements, union, intersection, De Morgan's laws, and so on using intuition. Unfortunately, when it comes to infinite sets, unless care is taken on exactly what kinds of collections of objects can be "accepted" as a set, it is possible to arrive at contradictions (i.e. Russell's paradox), which we will learn about later in this chapter. **Axiomatic** set theory was created to place set theory on a firm axiomatic foundation where the axioms are **consistent** (one can not prove contradictions) and **independent** (no one axiom can be proven from the others). The most accepted axioms of set

theory are the Zermelo-Fraenkel (ZF) axioms, named after logicians Ernst Zermelo (1871-1953) and Abraham Fraenkel (1891-1965).

Relation Between Sets and Logic

The properties of "$\cup$" and "$\cap$" in set theory have their counterparts in the properties of "$\vee$" and "$\wedge$" in sentential logic. Table 2 illustrates these counterparts.

Tautology	Set Equivalence
$P \vee Q \equiv Q \vee P$	$A \cup B = B \cup A$
$P \wedge Q \equiv Q \wedge P$	$A \cap B = B \cap A$
$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$	$A \cup (B \cup C) = (A \cup B) \cup C$
$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$	$A \cap (B \cap C) = (A \cap B) \cap C$
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$P \wedge P \equiv P$	$A \cap A = A$
$P \vee P \equiv P$	$A \cup A = A$

Equivalence between some Laws of Logic and Laws of Sets
Table 2

Theorem 4: **Complement Relation** For A and B sets, we have

$$A \subseteq B \Rightarrow \overline{B} \subseteq \overline{A}.$$

Proof:

The goal is to show $\overline{B} \subseteq \overline{A}$ with the aid of $\overline{B} \subseteq \overline{A}$. We have

let $x \in \overline{B}$

hence $x \notin B$ (definition of the compliment of B)

hence $x \notin A$ (since by hypotheses $A \subseteq B$)

hence $x \in \overline{A}$ (definition of the compliment of A)

Theorem 5: Distributivity Let A, B and C be sets. Then "$\cap$" distributes over "$\cup$". That is

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof: $(\subseteq)$ We prove $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$
Since

$$B \subseteq B \cup C$$
$$C \subseteq B \cup C$$

we intersect each side with A, getting

$$A \cap B \subseteq A \cap \left(B \cup C\right)$$
$$A \cap C \subseteq A \cap \left(B \cup C\right)$$

so $(A \cap B) \cup (A \cap C \subseteq A \cap (B \cup C)$.
$(\supseteq)$ We prove $A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$

If $x \in (A \cap B) \cup (A \cap C)$ we have $x \in A \cap B$ or $x \in A \cap C$. From the logical equivalence

$$\left[\left(p \wedge q\right) \vee \left(p \wedge r\right)\right] \equiv \left[p \wedge \left(q \vee r\right)\right]$$

this translates into set language as $x \in A$ and $x \in B \cup C$, or

$$x \in A \cap \left(B \cup C\right)$$

∎

De Morgan's Laws for Sets

In sentential logic, we were introduced to the important tautologies

$$\overline{\left(P \wedge Q\right)} \equiv \overline{P} \vee \overline{Q} \text{ and } \overline{\left(P \vee Q\right)} \equiv \overline{P} \wedge \overline{Q}$$

called De Morgan's laws. We now prove the set versions of these laws..

Theorem 6: De Morgan's Laws For sets *A* and *B*, prove De Morgan's laws:

$$a) \quad \overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$b) \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Proof

a) We prove the first De Morgan law by proving:

$(\subseteq)$ We let

$$\begin{aligned} x \in \overline{A \cup B} &\Rightarrow x \notin A \cup B \\ &\Rightarrow x \notin A \text{ and } x \notin B \\ &\Rightarrow x \in \overline{A} \text{ and } x \in \overline{B} \\ &\Rightarrow x \in \overline{A} \cap \overline{B} \end{aligned}$$

Hence, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

($\supseteq$) To show $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$, we let

$$\begin{aligned} x \in \overline{A} \cap \overline{B} &\Rightarrow x \in \overline{A} \text{ and } x \in \overline{B} \\ &\Rightarrow x \notin A \text{ and } x \notin B \\ &\Rightarrow x \notin A \cup B \\ &\Rightarrow x \in \overline{A \cup B} \end{aligned}$$

Combining the results of a) and b) we have $\overline{A \cup B} = \overline{A} \cap \overline{B}$. ▌
The second De Morgan law is left to the reader. See Problem 25.

We compare the set operations of union and intersection with the logical operations of "and" and "or," and the arithmetic operations of + and + and × in the following table.

Sets	Sentences	Arithmetic
$\cup$	$\vee$	$+$
$\cap$	$\wedge$	$\times$
$\subseteq$	$\Rightarrow$	$\leq$
$\overline{A}$	$\sim P$	$-$
$=$	$\equiv$	$=$
$\varnothing$	F	0
U	T	1

Problems

1. **Set Notation** Write the following sets in notation $\{x : P(x)\}$.

   a) The real numbers between 0 and 1.
   b) The natural numbers between 2 and 5.

2. **True or False** If $A = \{\{a\},\{b,c\},\{d,e,f\}\}$, tell if the following are true or false.

   a) $a \in A$ Ans: F
   b) $a \subseteq A$ Ans F
   c) $c \in A$
   d) $\{b,c\} \in A$
   e) $\varnothing \in A$
   f) $\varnothing \subseteq A$

3. **Checking Subsets: True or False**

a) $\mathbb{Z} \subseteq \mathbb{R}$ Ans: T

b) $\mathbb{R} \subseteq \mathbb{C}$ Ans: T

c) $(0,1) \subseteq [0,1]$ Ans: T

d) $(0,1) \subseteq \mathbb{R}$

e) $(2,5) \subseteq \mathbb{Q}$

f) $\mathbb{Q} \subseteq (2,5)$

g) $[1,3] \subseteq \{1,3\}$

h) $\{1,3\} \subseteq [1,3]$

i) $\{3,15\} \subseteq \{3,5,7,15\}$

4. **The Empty Set: True or False**

a) $\varnothing = \{\varnothing\}$ Ans: F

b) $\varnothing \in \{\varnothing\}$ Ans: T

c) $\varnothing \subseteq \{\varnothing\}$

d) $A \cup \varnothing = A$

e) $\{\varnothing\} \subseteq \varnothing$

f) $\{\varnothing\} \in \{\{\varnothing\}\}$

g) $\{\{\varnothing\}\} \in \{\varnothing, \{\varnothing\}\}$

5. **True or False**

a) $A \in A$ Ans: F

b) If $A \subseteq B$ and $x \notin B$ then $x \notin A$ Ans: T

c) $A \subseteq B$ then $A \in B$.

d) $A \in B$ then $A \subseteq B$

e) $A \in B$ and $B \in C$ then $A \in C$

f) $A \in B$ and $B \in C$ then $A \subseteq C$

6. **Sets, Members and Subsets** Fill in the blank with one of the combinations $(\in, \subseteq)$, $(\in, \not\subseteq)$, $(\notin, \subseteq)$, $(\notin, \not\subseteq)$ that describe the given relation.

a) a ___ $\{c, a, t\}$ Ans: $\in$, $\not\subseteq$

b) a ___ $\{c, a, \{a\}, t\}$ Ans: $\in$, $\not\subseteq$

c) $\{a,t\}$ ___ $\{c,a,t\}$ Ans: $\notin, \subseteq$
d) $\{a\}$ ___ $\{c,\{a\},t\}$
e) $\{a,\{t\}\}$ ___ $\{c,a,t,\{t\}\}$
f) $\{a,\{t\}\}$ ___ $\{c,a,t,\{t\},\{a,\{t\}\}\}$
g) $\{c,\ a,\{t\}\}$ ___ $\{a,t,\{t\}\}$
h) $\{a,\{t\}\}$ ___ $\{c,a,t,\{t\}\}$
i) $\varnothing$ ___ $\varnothing$
j) $\{\varnothing\}$ ___ $\{\varnothing\}$
k) $\{\varnothing\}$ ___ $\{\varnothing,\{\varnothing\}\}$
l) $\varnothing$ ___ $\{\varnothing\}$
m) $\{\varnothing\}$ ___ $\varnothing$

7. **Power Sets** Find the power set of the given sets.

a) $A=\{4,5,6\}$
b) $A=\{\oplus,\odot,\otimes\}$
c) $A=\{a,\{b\}\}$
d) $A=\{a,\{b,\{c\}\}\}$
e) $A=\{a,\{a\}\}$
f) $A=\{\varnothing,\{\varnothing\}\}$

8. **Matching Sets** Which pairs of the following sets are connected by one or more of the relations $\in,\subseteq,$ or $\supseteq$?

a) $\mathbb{R}$
b) 3
c) $\{1,2,3,...,10\}$
d) $\{x : x \text{ is an even integer}\}$
e) $(-1,1)$
f) $\varnothing$

9. **Find the Set** Let $A=\{a_1,a_2,..\}$ where a_n is the remainder when n is divided by 5. List the elements of A.

10. **Interesting** If $A=\{a,b,c\}$ are either of the following true?

a) $A \in P(A)$
b) $A \subseteq P(A)$

11. **Power Set as a Collection of Functions** The power set of a set can be interpreted as the set of all functions[6] defined on the set whose values are 0 and 1. For example, the functions defined on $A=\{a,b\}$ with values 0 and 1 are

- $f(a)=0, f(b)=0$ corresponds to $\varnothing$
- $f(a)=1, f(b)=0$ corresponds to $\{a\}$
- $f(a)=0, f(b)=1$ corresponds to $\{b\}$
- $f(a)=1, f(b)=1$ corresponds to $\{a,b\}$

Show that the elements of the power set of $A=\{a,b,c\}$ can be placed in this one-to-one correspondence with the functions on A whose values are either 0 or 1.

12. **Second Power Set** The second power set of a set A is the set of subsets of the set of subsets of the set, or $P(P(A))$. What is the second power set of $A=\{a\}$?

13. **Power Set of the Empty Set** Prove $P(\varnothing)=\{\varnothing\}$.

14 **The Man Who Constructed Something from Nothing** The German mathematician Leopold Kronecker once said,*"God created the integers, all else is the work of man.*" But logicians like to say that logicians created the integers from more basic axioms. The German logician Gottlieb Frege defined the integers from nothing; starting with the *empty set!* How did he do it? He defined the non-negative integers recursively:

$$0=\varnothing$$
$$1=\{0\}=\{\varnothing\}$$
$$2=\{0,1\}=\{\varnothing,\{\varnothing\}\}$$
$$3=\{0,1,2\}=\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$$
$$\cdots \qquad \cdots \qquad \cdots$$

[6] Although we haven't introduced functions yet in this book, we are confident most readers have familiarity with the subject.

and so on. Any thoughts on how you would construct an arithmetic using this definition. How would you define "1 + 1" so you would get 2? What about "1 + 3"?

15. **Identities** Let A, B and C be arbitrary subsets of some universe U . Prove the following.

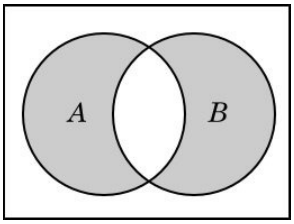
a) $A \subseteq A$
b) $A \cup \varnothing = A$
c) $A \cap \varnothing = \varnothing$
d) $\varnothing = \overline{U}$
e) $A \cap U = A$
f) $A \cap \overline{A} = \varnothing$
g) $A \subseteq B \Rightarrow A \cup B = B$
h) $A \cup A = A \cap A$
i) $\overline{\overline{A}} = A$

16. **Difference Between Sets** The formula $A - B = A \cap \overline{B}$ defines the difference between two sets in terms of the intersection and complement. Can you find a formula for the union $A \cup B$ in terms of intersections and complements?

17. **Symmetric Difference** The **symmetric difference** to two sets A and B is the set of elements that belong to one of the sets but not both and is denoted by

$$A \Delta B = (A - B) \cup (B - A).$$

It is both commutative and associative and is analogous to the exclusive OR in logic. See Figure 4.



$A \Delta B$
Figure 4

Given the sets $E = \{2,4,6,\dots\}$ and $O = \{1,3,5,\dots\}$ find the symmetric differences.

a) $E \Delta O$
b) $E \Delta \varnothing$
c) $E \Delta E$

18. **NASC for Disjoint Sets** Prove $A \cap B = \varnothing \Leftrightarrow A - B = A$.

19. **Distributive Law** Prove if A, B and C are sets, then "$\cup$" distributes over "$\cap$". That is

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

20. **Set Identity** Prove $A \subseteq B \Leftrightarrow A \cup B = B$.

21**. Proving Set Relations with Truth Tables** It is possible to prove identities of sets using truth tables. For example, one of DeMorgan's laws

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

is verified from the following truth table, replacing the union by "or," the intersection by "and," and set complementation by "not."

		(1)	(2)	(3)	(4)	(5)
$x \in A$	$x \in B$	$A \cup B$	$\overline{(A \cup B)}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
T	T	T	**F**	F	F	**F**
T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

same truth values

Prove the following identities using truth tables.

a) $A \cap \overline{A} = \varnothing$
b) $A \cup \overline{A} = U$
c) $A \cup (B \cup C) = (A \cup B) \cup C$
d) $A \cup B = (A \cap B) \cup (A - B) \cup (B - A)$

22. **Sets and Their Power Sets** Prove

$$A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$$

23. **Relationship Between Logic and Set Operations** Using the fact that an implication in sentential logic is equivalent to its contrapositive, show that $A \subseteq B \Leftrightarrow \overline{B} \subseteq \overline{A}$.

24. **Computer Representation of Sets** Finite sets can be represented efficiently by vectors of 0s and 1. Consider the set $U = \{a_1, a_2, \ldots, a_n\}$. We can represent a subset S of this set by a bit string, where the i^{th} bit is 1 if $a_i \in S$ and 0 if $a_i \notin S$. The following problems relate to this representation of sets. Take as the universe the set $U = \{0,1,2,3,4,5,6,7\}$.

a) If $U = \{3,9,2,5,6\}$, what is $S \subseteq U$ for the bit string 11001 ?

b) If $U = \{1,2,3,4,5,6\}$ what is the bit string for $A = \{2,6\}$?

c) If $U = \{1,2,3,4,5,6\}$ and $S = \{1,4,5\}, T = \{1,2,4,6\}$ what is the bit string for $S \cap T$?

d) If $U = \{1,2,3,4,5,6\}$ and $S = \{1,4,5\}, T = \{1,2,4,6\}$ what is the bit string for $S \cup T$?

e) How would you represent the complement of a subset of $U = \{1,2,3,4,5,6\}$?

25. **De Morgan's Law** Prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

ΓΣΘΨΞΠΩ