Section 2.2 Families of Sets

Purpose of Section: We now extend the set operations of union and intersection from two sets to several sets, even an infinite number of sets.

Introduction

The union and intersection of sets can be extended readily to the union and intersection of many, even an infinite number of sets. When dealing with a collection of several sets, it is common practice to refer to them as **families** or **classes** of sets.. To denote families of sets, one often uses indices such as $\{A_1, A_2, A_3, \cdots, A_{10}\}$ For an infinite family, we might write $\{A_1, A_2, A_3, \cdots\}$ or maybe

$$\{A_k\}_{k=1}^{\infty}.$$

Other common ways to denote families of sets are

$$\{A_i : i \in \Lambda\} \text{ or } \{A_k\}_{k\in\Lambda}$$

where the set Λ is called an **index set**. For example,

$$I_n = \left\{\left[0, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$$

denotes the infinite family of intervals

$$\left\{[0,1), \left[0, \frac{1}{2}\right), \left[0, \frac{1}{3}\right), \ldots\right\}.$$

The reader might recall the notation for infinite sums and products to be

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots, \qquad \prod_{k=1}^{\infty} a_k = a_1 a_2 \cdots$$

which motivates the following notation and definition.

Important Note: We have introduced five number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. We know that $\mathbb{N}$ stands for natural numbers, $\mathbb{Q}$ for quotients, $\mathbb{R}$ for real numbers, and $\mathbb{C}$ for complex numbers. But where does the letter $\mathbb{Z}$ which represents the integers come from? The answer is that $\mathbb{Z}$ refers to the first letter of ‘Zahl’, the German word for number.

Definition : Unions and Intersections of Families of Sets The union of the family of subsets $A_1, A_2, \cdots, A_n$ of a universe U is defined as

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{k=1}^{n} A_k = \{x \in U : x \in A_k \text{ for some } k = 1, 2, ..., n\}$$

The **intersection** of the sets $A_1, A_2, \cdots, A_n$ is

$$A_1 \cap A_2 \cap A_3 \cdots \cap A_n = \bigcap_{k=1}^{n} A_k = \{x \in U : x \in A_k \text{ for all } k = 1, 2, ..., n\}$$

Sometimes one has a more general families of sets $\{A_\alpha\}_{\alpha \in \Lambda}$ where the index α ranges over some **index set** Λ (Λ could be a finite set, the natural numbers, or all values in an interval on the real line.). In this case the **union of the family** $\{A_\alpha\}_{\alpha \in \Lambda}$ is

$$\bigcup_{\alpha \in \Lambda} A_\alpha = \{x \in U : x \in A_\alpha \text{ for at least one } \alpha \in \Lambda\}$$

and the **intersection of the family** $\{A_\alpha\}$ is

$$\bigcap_{\alpha \in \Lambda} A_\alpha = \{x \in U : x \in A_\alpha \text{ for all } \alpha \in \Lambda\}$$

The following examples illustrate these ideas.

Example 1: Infinite Intersections and Unions Define the family of closed intervals

$$A_k = \left[0, \frac{k-1}{k}\right], \; k = 2, 3, ...$$

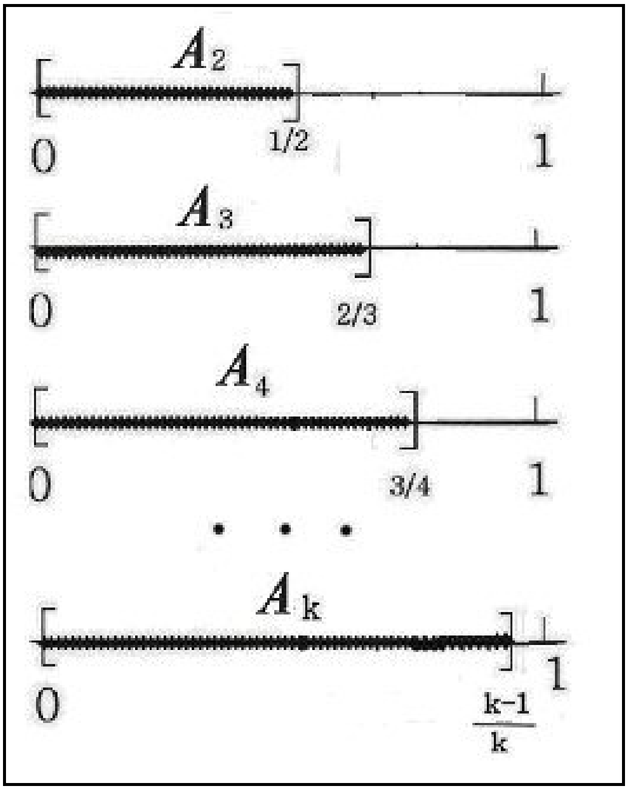
where a few are drawn in Figure 1. Find the following unions and intersections.

a) $\bigcup_{k=2}^{4} A_k$ b) $\bigcup_{k=2}^{\infty} A_k$

c) $\bigcap_{k=2}^{4} A_k$ d) $\bigcap_{k=2}^{\infty} A_k$

Solution

The sets $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ form an increasing family of closed intervals, each member in the family is a subset of the next.



Increasing family of closed intervals
Figure 1

a) $\bigcup_{k=2}^{4} A_k = \left[0, \frac{3}{4}\right]$

b) $\bigcup_{k=2}^{\infty} A_k = [0,1)$

c) $\bigcap_{k=2}^{4} A_k = \left[0, \frac{1}{2}\right]$

d) $\bigcap_{k=1}^{\infty} A_k = \left[0, \frac{1}{2}\right]$

Important Note Families of sets are commonplace in the real world, too. The collection of classes at a university is a family of sets, where each class is a set of students.

Example 2: Infinite Intersections of Unions Define the sequence of open intervals

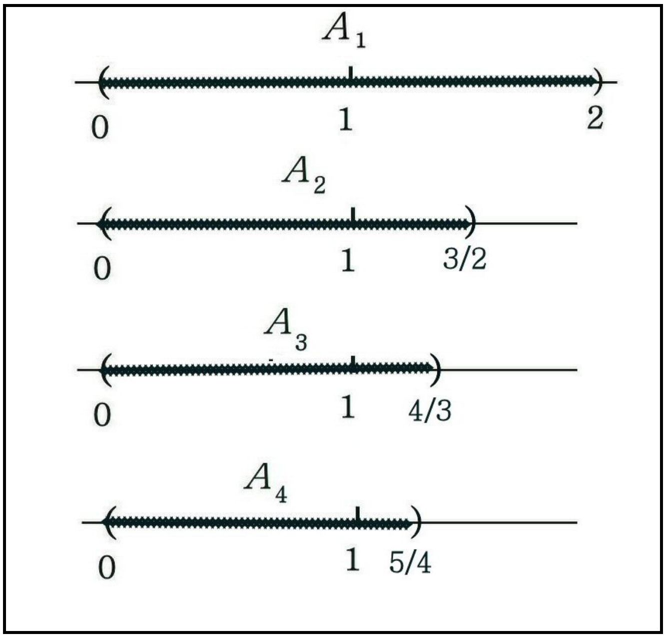
$$A_k = \left(0, \frac{k+1}{k}\right), \ k = 1, 2, ...$$

Find the following union and intersection.

$$a) \ \bigcup_{k=1}^{\infty} A_k = A_1 \cup A_2 \cup \cdots \qquad b) \ \bigcap_{k=1}^{\infty} A_k = A_1 \cap A_2 \bigcap \cdots$$

Solution

A few sets in the family are drawn in Figure 2.



Decreasing family of open intervals
Figure 2

We find

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty}\left(0, \frac{k+1}{k}\right) = \left(0, \frac{2}{1}\right) \cup \left(0, \frac{3}{2}\right) \cup \left(0, \frac{4}{3}\right) \cup \cdots = (0, 2)$$

$$\bigcap_{k=1}^{\infty} A_k = \bigcap_{k=1}^{\infty}\left(0, \frac{k+1}{k}\right) = \left(0, \frac{2}{1}\right) \cap \left(0, \frac{3}{2}\right) \cap \left(0, \frac{4}{3}\right) \cap \cdots = (0, 1]$$

Example 3: Indexed Family Define a sequence of families of sets by

$$A_k = \{k+1, k+2, \cdots, 2k\} \ , \ \ k = 1, 2, \ldots$$

Do you understand that

$$\bigcup_{k=1}^{\infty} A_k = \{n \in N : n \geq 2\} \qquad \bigcap_{k=1}^{\infty} A_k = \varnothing$$

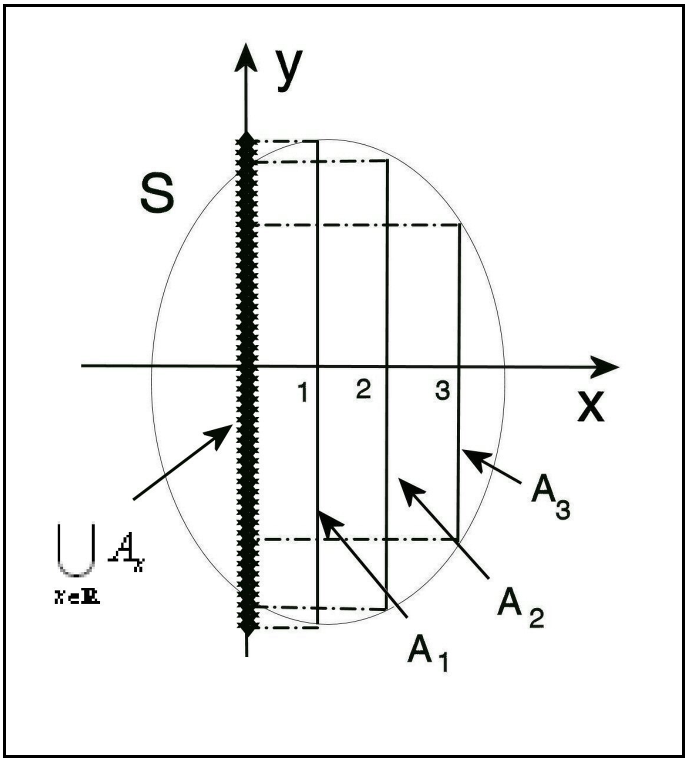
Example 4: Set Projection Let $S \subseteq \{(x,y) : x, y \in \mathbb{R}\}$ denote a subset of the plane. For each real number x define the set

$$A_x = \{y \in \mathbb{R} : (x,y) \in S\}$$

For each $x \in \mathbb{R}$ this set defines those values of y such that $(x,y) \in S$. The union

$$\bigcup_{x \in \mathbb{R}} A_x$$

is the projection of S onto the y-axis as illustrated in Figure 3.



Projection of a Set
Figure 3

Extended Laws for Sets

Many of the laws for intersection and union of sets that we studied in Section 2.1 can be extended to families of sets. We leave the proofs to most of these laws to the reader.

Laws for Families of Sets

a) $A \cap \left(\bigcup_{\alpha\in\Lambda} B_\alpha \right) = \bigcup_{\alpha\in\Lambda} (A \cap B_\alpha)$

b) $A \cup \left(\bigcap_{\alpha\in\Lambda} B_\alpha \right) = \bigcap_{\alpha\in\Lambda} (A \cup B_\alpha)$

c) $\overline{\bigcap_{\alpha\in\Lambda} A_\alpha} = \bigcup_{\alpha\in\Lambda} \overline{A_\alpha}$ (De Morgan's Law)

d) $\overline{\bigcup_{\alpha\in\Lambda} A_\alpha} = \bigcap_{\alpha\in\Lambda} \overline{A_\alpha}$ (De Morgan's Law)

Proof of d) We first show $\overline{\bigcup_{\alpha\in\Lambda} A_\alpha} \subseteq \bigcap_{\alpha\in\Lambda} \overline{A_\alpha}$ by letting

$$
\begin{aligned}
x \in \overline{\bigcup_{\alpha\in\Lambda} A_\alpha} \Rightarrow\ & x \notin \bigcup_{\alpha\in\Lambda} A_\alpha \\
&\Rightarrow (\forall \alpha \in \Lambda)(x \notin A_\alpha) \\
&\Rightarrow (\forall \alpha \in \Lambda)(x \in \overline{A}_\alpha) \\
&\Rightarrow (\forall \alpha \in \Lambda)\left(x \in \bigcap_{\alpha\in\Lambda} \overline{A_\alpha} \right)
\end{aligned}
$$

Hence,

$$\overline{\bigcup_{\alpha\in\Lambda} A_\alpha} \subseteq \bigcap_{\alpha\in\Lambda} \overline{A_\alpha} .$$

/The proof of the set containment $\supseteq$ follows along the same lines and is left to the reader.

Topologies on a Set

A topology on a set is a family of subsets of the set that allow for the study of convergence of points in the set. The study of point-set topology forms the basis of many areas of mathematics such as topology, and real and complex analysis.

The idea is to introduce a family J of subsets of a given set U. A typical set U would be the real numbers or the Cartesian plane. The sets in the family J are called **open sets** and these sets act as "neighborhoods" of points in U, allowing for the discussion of convergence sequences in U. This family of open sets J is called a **topology** on the set U. But not any collection of subsets of U

is called a topology? There are three restrictions on a family of subsets of U in order that it forms a topology on U. They are as follows:

Definition: A **topology** J on a set U is any collection of subsets of U that satisfies the following conditions:

1) The empty set $\varnothing$ and U belong to the family J.
2) The *union* of any collection of sets in J also belongs to J.
3) The *intersection* of any finite number sets in J also to J.

The sets in the topology J are called the open **sets in the topology** (or just **open sets**). Properties 2) and 3) say that the family J is **closed** under **unions** and **finite intersections**.

As an illustration, consider the set U

$$U = \{a,b,c\}$$

So what subsets of U can we choose as our open sets? We know U has a total of $2^3 = 8$ subsets. Below we list five families of subsets of U, each of which satisfies the given conditions of being a topology on U. The reader can verify (See Problem 8) each family J_1, J_2, J_3, J_4, J_5 satisfies the conditions for being a topology on U. The topology J_1 contains two open sets, the empty set $\varnothing$ and the universe U, and is called the **indiscrete topology** for U. At the other extreme, the topology J_5 contains all subsets of U and is called the **discrete topology**, which means that every subset of U is an open set. The other topologies J_2, J_3, J_4 are between the two extreme topologies.

$J_1 = \{\varnothing, U\}$	indiscrete topology
$J_2 = \{\varnothing, \{a\}, U\}$	
$J_3 = \{\varnothing, \{a\}, \{b,c\}, U\}$	
$J_4 = \{\varnothing, \{a,b\}, U\}$	
$J_5 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, U\}$	discrete topology

Theorem 1 Let $U = \{a,b,c\}$. The power set

$$J = \{\varnothing, \{a\}, \{b,c\}, \{a,b,c\}\} = P(U)$$

of U is a topology on U.

Proof: We verify the three conditions required for a topology. The first condition is verified since the topology J contains both the empty set $\varnothing$ and U. To verify that J is closed under unions, we take the unions of sets in J and see the that the union also belongs to J For example,

$$\{a\} \cup \{b,c\} = \{a,b,c\} \in J$$
$$\varnothing \cup \{a\} = \{a\} \in J$$
$$\{a\} \cup \{a,b,c\} = \{a,b,c\} \in J$$

To show the family J is closed under intersections, we take intersections of members of J and observe their intersections also belong to J. For example,

$$\{a\} \cap \{b,c\} = \varnothing \in J$$
$$\{a\} \cap \{b,c\} = \{a\} \in J$$
$$\{b,c\} \cap \{a,b,c\} = \{b,c\} \in J$$

So what do these topologies on U have to do with convergence? We will discover how in Section 5.4 when we introduce point-set topology and a neighborhood of a point.

Problems

1.Let $A_k = \{k, k+1\}$, $k = 1,2,...$ denote a sequence of pairs of positive integers.. Describe the following sets.

a) $\bigcup_{k=1}^{5} A_k$ Ans: $\{1,2,3,4,5,6\}$

b) $\bigcup_{k \in \mathbb{N}} A_k$ Ans: $\mathbb{N}$

c) $\bigcup_{k \geq 5} A_k$

d) $\bigcup_{1 \leq k \leq 4} A_k$

e) $\bigcap_{k=1}^{5} A_k$

f) $\bigcap_{k \in \mathbb{N}} A_k$

2. **Unions and Intersections** Find the infinite union and intersections

$$\bigcup_{k=1}^{\infty} A_k \text{ and } \bigcap_{k=1}^{\infty} A_k$$

for the following sets.

a) $A_k = \left[0, \frac{k-1}{k}\right]$

b) $A_k = \left[-\frac{1}{k}, \frac{1}{k}\right]$

c) $A_k = \left(0, \frac{1}{k}\right)$

d) $A_k = \{k\} \cup \left[\frac{1}{k}, 2k\right]$

e) $A_k = [k, k+1]$

f) $A_k = \left[0, 1+\frac{1}{k}\right]$

3. **Families of Sets in the Plan**e Define a family of sets in the plane $\mathbb{R}^2$ by $A_{m,n} = \{(x, y) \in \mathbb{R}^2 : x \geq m, y \geq n\}$ where $a, b \in \mathbb{R}$. Find the following sets. Hint: Proceed exactly like one does with double series.

a) $\bigcup_{n=1}^{3}\left(\bigcup_{m=2}^{3} A_{a,b}\right)$

b) $\bigcup_{n=1}^{3}\left(\bigcap_{m=2}^{3} A_{a,b}\right)$

4. **Identity for an Indexed Family** Show

$$B \cap \left(\bigcup_{\alpha \in \Lambda} A_\alpha\right) = \bigcup_{\alpha \in \Lambda}\left(B \cap A_\alpha\right)$$

5. **Algebra of Sets** Let A be a set and $\Im$ a collection of subsets of A. The collection $\Im$ is called an **algebra**[1] of sets if

a) $A \cup B$ is in $\Im$ whenever A and B are in $\Im$
b) $\overline{A}$ is in $\Im$ whenever A is in $\Im$

When this happens we say the family $\Im$ is **closed** under **unions** and **complementation**. Which of the following collections of subsets of $A = \{a,b,c\}$ constitute an algebra of subsets of A**?**

a) The power set $\Im = P(A)$
b) $\Im = \{\varnothing, A\}$
c) $\Im = \{\varnothing, \{a\}, A\}$
d) $\Im = \{\varnothing, \{a\}, \{b,c\}, A\}$

6. **Sets of Length Zero** In measure theory, a subset A of the real numbers is said to have **length** (or **measure**) zero if $\forall \varepsilon > 0$ there exists a sequence $A_k = (a_k, b_k)$ of open intervals that "cover" A: i.e.

$$A \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k)$$

and their *total length* is less than ε; that is,

$$\sum_{k=1}^{\infty} |b_k - a_k| < \varepsilon \ .$$

Show that any sequence of real numbers $\{c_k\}, k = 1,2,...$ has measure zero. Hint: Cover each element c_k in the sequence by an interval (a_k, b_k) of length $|b_k - a_k| = \varepsilon / 2^k$.

7. **Compact Sets** A subset A of the real numbers is said to be **compact** if for every collection $\Im = \{(a_\alpha, b_\alpha) : \alpha \in \Lambda\}$ of open intervals that contains (or covers) A; i.e.

$$A \subseteq \bigcup_{\alpha \in \Lambda} (a_\alpha, b_\alpha)$$

there exists a *finite* subcollection of intervals of $\Im$ whose union also contains (covers) A. Show the set $A = (0,1)$ is not compact by showing the following:

[1] Algebras of sets and sigma algebras (families of sets closed under *countable* unions) are fundamental in the study of measure theory. Note the difference between an algebra of subsets and a topology of subsets on a universe; just a minor difference makes for vastly different structures on the universe.

a) A is covered by

$$\Im = \left\{ \left(0, 1 - \frac{1}{k} \right) : k = 1, 2, ... \right\}$$

b) There does not exist a finite sub-collection of $\Im$ whose union contains A.

8. **Topologies** Verify that the following families are topologies on $\{a, b, c\}$.

$$J_1 = \{\varnothing, U\}$$
$$J_2 = \{\varnothing, \{a\}, U\}$$
$$J_3 = \{\varnothing, \{a\}, \{b, c\}, U\}$$
$$J_4 = \{\varnothing, \{a, b\}, U\}$$
$$J_5 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, U\}$$

9. **Topologies** Which of the following families of subsets of $\{a, b, c\}$ are topologies on $\{a, b, c\}$?

a) $J = \{\varnothing, \{b\}, \{c\}, \{a, b, c\}\}$
b) $J = \{\varnothing, \{a, b\}, \{b, c\}, \{a, b, c\}\}$
c) $J = \{\varnothing, \{a\}, \{b, c\}\}$
d) $J = \{\varnothing, \{b\}, \{b, c\}, \{a, b, c\}\}$

10. **Finding Intersections** Find an infinite family of sets, none of which is the given set, whose intersection is

a) $\{1\}$

b) $[0, \infty)$

11. **Finding Unions** Find an infinite family of sets, none of which is the given set, whose union is

a) $(0, \infty)$
b) $\mathbb{R}$

12. **Quotient Set** $\mathbb{Z}/5$ Given the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ we say that two integers are equivalent (mod 5) if they have the same remainder when divided by 5. Recalling that a negative number like -4 has a remainder of 1 when divided by 5 since

$$\frac{-4}{5} = \frac{-5+1}{5} = -1 + \frac{1}{5}$$

find the family of sets where each set in the family consists of sets of equivalent integers. This family of equivalent sets is an example of a **quotient set**, which in this case we denote by $\mathbb{Z}/5$.

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