Section 2.3 Counting: The Art of Enumeration

Purpose of Section To introduce some basic tools of counting such as the multiplication principle, permutations and combinations. We show how the basic multiplication rule gives rise to counting permutations and combinations. We close with a brief introduction to the pigeonhole principle, and illustrate how it can be applied to solving a certain class of problems.

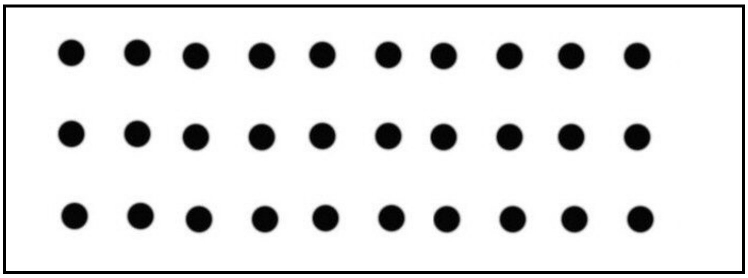
Introduction Although counting is one of the first things we learn at a tender age, be assured there are counting problems that test the ability of the brightest among us. In this section, we will learn to count, although we call it *enumerative combinatorics.* We will answer such counting questions as, *how many different ways can 12 people be seated at two tables with one seating 7 and the other seating 5?*

Counting problems often require little technical background and are characterized by being easy to understand and hard to solve. Finding the number of ways to cover an 8 x 8 checkerboard with dominoes, where each domino covers two adjacent squares, is a good example. The problem is easy to understand, but the number of coverings was not determined until 1961 by a M. E. Fischer, who found the number to be $2^4 \times (901)^2 = 12,988,816$.

To assist you in perfecting your counting skills, we introduce one of the basic tools of the trade, the multiplication principle.

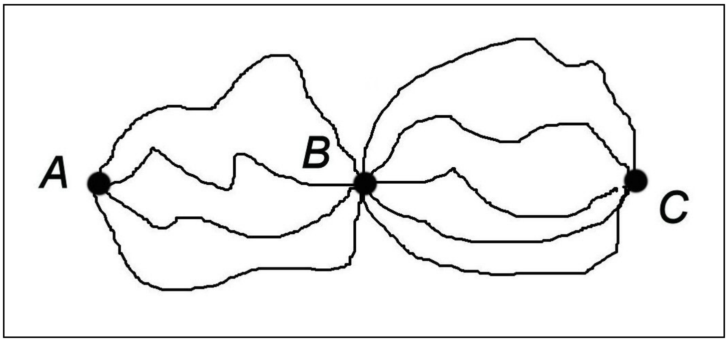
Multiplication Principle

One of the most basic principles of counting is the **multiplication principle**. Although the principle is simple, it has far reaching consequences. For example, how many dots are there in following array?



We suspect you said 30 after about three seconds, and you didn't even bother to count all the dots. You simply counted the number of columns and multiplied by 3. If so, you used the multiplication principle.

As a small step up the complexity chain, try counting the number of different paths from A to C in Figure 1. No doubt this problem didn't stump you either, getting $4 \cdot 5 = 20$ paths. Again, you used the multiplication principle.



How Many Paths from *A* to *C*?
Figure 1

This leads us to the formal statement of the multiplication principle.

> **Multiplication Rule**: If a procedure can be broken into successive stages, and if there are s_1 outcomes for the first stage, s_2 outcomes for the second stage, …, and s_n for the nth stage, then the entire procedure has $s_1 s_2 ..., s_n$ outcomes.

We now let you test your counting skills using the multiplication rule.

Example 1: Counting Subsets Show that a set $A = \{a_1, a_2, ..., a_n\}$ containing n elements has 2^n subsets.

Proof: A subset of A can be formed in n successive steps. On the first step, pick or not pick a_1, on the second step pick or not pick a_2, and so on. On each step, there are two options, to pick or not to pick. Using the multiplication principle, the number of subsets that can picked is

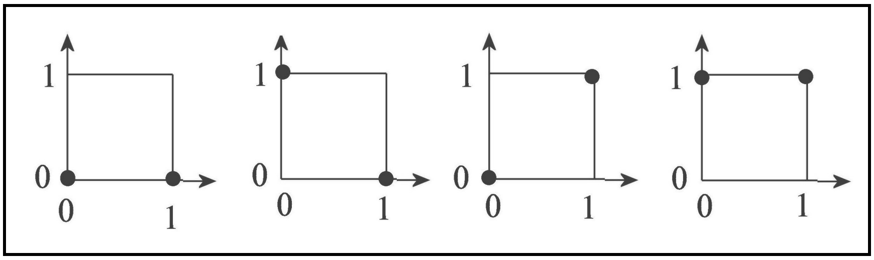
$$\underbrace{2 \cdot 2 \cdot ... \cdot 2}_{n \text{ twos}} = 2^n$$

Hence, the set $A = \{a, b, c\}$ containing three members has $2^3 = 8$ the eight subsets:

$$P(A) = \{\varnothing . \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

Example 2: Counting Functions[1] How many functions are there from the set $A = \{0,1\}$ to itself?

Solution For each of the 2 members in A, the function can take on 2 values and so by the multiplication rule, the number of functions is $2 \cdot 2 = 2^2 = 4$. We graph these functions in Figure 2.



Four functions from $\{0,1\}$ to $\{0,1\}$

Figure 2

In general, the number of functions from a set with n members to a set with m members is m^n.

Permutations

One of the important uses of the multiplication principle is counting permutations. A **permutation** is an **arrangement** of objects. For example, the three letters abc have six permutations, which are

$$abc, acb, bac, bca, cab, cba$$

The number of permutations increases dramatically as the size of the set increases. The number of permutations of the first 10 letters of the alphabet $abcdefghij$ is $3,628,800$. To determine the number of arrangements, we select the left-most member in our arrangement, which can be done in 10 ways.. Once this is done, there are 9 choices for the second member in the arrangement, the third member 8 ways, and so on down to the last member. Using the multiplication principle, we find the number of permutations or arrangements of 10 objects to be 10 factorial, or

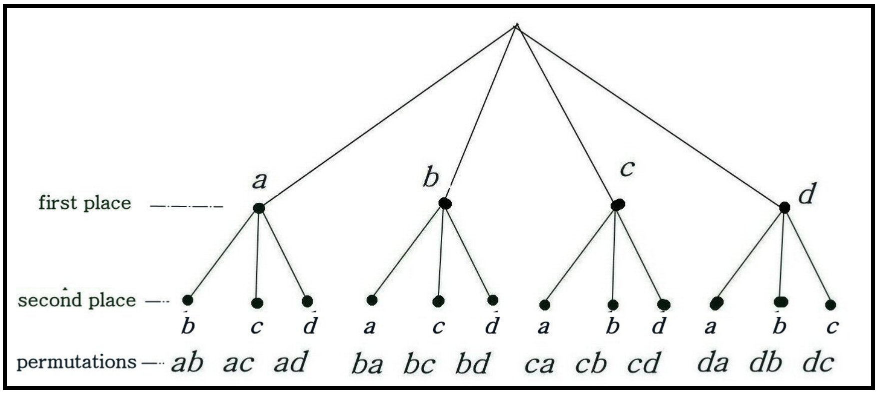
$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3,628,800$$

Permutations of Racers

Suppose four individuals a, b, c, d are in a foot race and we wish to determine the possible ways the runners can finish first and second. Each of the 4 runners can finish first, and for each winner, there are 3 second-place finishers. The

[1] We will talk more about functions in Chapter 4.

multiplication principle give the number of possible ways to be $4 \cdot 3 = 12$, The tree diagram in Figure 3 illustrates these finishers.



Permutations of two elements from a set of four elements
Figure 3

This leads to the general definition of permutations of different sizes.

Definition: A **permutation** of r elements taken from a set of n elements is an arrangement of r elements chosen from a set of n elements. The number of such arrangements (or permutations) is denoted by $P(n,r)$.

Using the multiplication principle, we find the number of permutations.

Theorem 1 Number of Permutations The number of permutations of r elements taken from a set of n elements is

$$P(n,r) = \frac{n!}{(n-r)!} = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

Proof:

Choosing r elements from a set of size n, we have

- the first element can be selected n ways.
- the second element can be selected $n-1$ ways (since now there are $n-1$ left).
- the third element can be selected $n-2$ ways (since now there are $n-2$ left).

... ...

- the rth element can be selected $n-r+1$ ways

Hence, by the principle of sequential counting (or the multiplication rule), we have

$$P(n,r)=\underbrace{n\cdot(n-1)\cdot(n-2)\cdots(n-r+1)}_{r\text{ factors}}.$$

When $r=n$ we have the number of permutations of n objects to be n factorial, or

$$P(n,n)=n(n-1)(n-2)\cdots(2)(1)=n!$$

Important Note: To evaluate $P(n,r)$ start at n and multiply r factors.

$$P(4,2)=4\cdot 3=12$$
$$P(7,3)=7\cdot 6\cdot 5=210$$
$$P(4,1)=4=4$$
$$P(10,3)=10\cdot 9\cdot 8=720$$

Big Number: Factorials grow very fast. The number $50!$ is *thirty vigintillion, four hundred and four novemdecillion, ninety three octodecillion, two hundred and one septendecillion, seven hundred and thirteen sexdecillion, three hundred and seventy eight quindecillion, forty three quatuordecillion, six hundred and twelve tredecillion, six hundred and eight dodecillion, one hundred and sixty six undecillion, sixty four decillion, seven hundred and sixty eight nonillion, eight hundred and forty four octillion, three hundred and seventy seven septillion, six hundred and forty one sextillion, five hundred and sixty eight quintillion, nine hundred and sixty quadrillion, five hundred and twelve trillion*. That is

$$50!=3040409320171337804361260816606476884437764156896051200000000000$$

Example 3: Permutations of a Set with 3 Elements Find the permutations of size $r = 1, 2$ and 3 selected from $\{a, b, c\}$.

Solution

The permutations of size $r = 1, 2, 3$ taken from the set $\{a, b, c\}$ with $n = 3$ elements are listed in Table 1.

$r=1$	$r=2$	$r=3$
a	ab	abc
b	ac	acb
c	ba	bac
	bc	bca
	ca	cab
	cb	cba

Permutations $P(n, r)$

Table 1

We don't use set notation for writing permutations[2] since order is important. The permutation ab is not the same as ba.

Example 4: Indistinguishable Permutations How many ways can one arrange the seven letters of the word SYSTEM?

Solution

Since two S's in SYSTEM are indistinguishable[3] we must modify our argument. If we momentarily call the two S's as S_1 and S_2, then we have $\mathrm{S_1YS_2TEM}$, which has 6! permutations. But the two letters S_1 and S_2 have two permutations, so we must divide 6! by 2!, getting the number of *distinguishable* (permutations which you can recognize with the naked eye) is

$$P(6,4) = \frac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$

A few more distinguishable permutations are given in the following table.

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[2] Sometimes permutations are written with round parenthesis, such as $ab = (ab)$.

[3] If we interchanged the two S's, we would get two different permutations, SYSTEM and SYSTEM, but they are not *distinguishable* permutations since the S's look alike.

Word	Indistinguishable Permutations
too	$\frac{3!}{2!}=3$
error	$\frac{5!}{3!}=20$
toot	$\frac{4!}{2!2!}=6$
Mississippi	$\frac{11!}{4!4!2!}=34,650$

Distinguishable permutations
Table 2

Combinations

Combinations are simply subsets of a given set. For the set $\{a,b,c\}$ of three members there are three combinations (or subsets) of size 1, three combinations (or subsets) of size 2, and 1 combination (or subset) of size 3. The three combinations of size two are $\{a,c\},\{a,t\},\{c,t\}$.

We write combinations in set notation $\{\ \}$ since the order of members in the combination does not matter. The combination $\{a,c\}$ is the same as the combination $\{c,a\}$.This is the major difference from permutations where the permutation ab is not the same as the permutation ba.

Example 5: Combinations Find the combinations of size $r=1,2$ and 3 selected from the set $\{a,b,c\}$.

Solution

The combinations are listed Table 3, which are the 8 subsets of $\{a,b,c\}$ with the exception of the empty set $\varnothing$.

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$r=1$	$r=2$	$r=3$
$\{a\}$	$\{a,b\}$	$\{a,b,c\}$
$\{b\}$	$\{a,c\}$	
$\{c\}$	$\{b,c\}$	

Nonempty subsets of $\{a,b,c\}$
Table 3

Theorem 2 Number of Combinations The number of **combinations** of size r is taken from a set of size n is

$$C(n,r)=\binom{n}{r}=\frac{n!}{r!(n-r)!}.$$

Proof Since the number of permutations of size r taken from a set of size n is

$$P(n,r)=\frac{n!}{(n-r)!}.$$

and since r elements can be permuted $r!$ways, we divide the number of permutations by $r!$ getting

$$C(n,r)=\frac{P(n,r)}{r!}=\frac{n!}{r!(n-r)!}.$$

We denote combinations by either notation

$$C(n,r) \text{ or } \binom{n}{r}$$

and are possibly familiar to the reader since they are the coefficients in the binomial expansion formulas

$$(a+b)^2=\binom{2}{0}a^2+\binom{2}{1}ab+\binom{2}{2}b^2=a^2+2ab+b^2$$

$$(a+b)^3=\binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3=a^3+3a^2b+3ab^2+b^3$$

$$\cdots \quad \cdots \quad \cdots$$

It helps in thinking about combinations to read $C(n,r)$ as "n *choose* r"since it represents the number of ways you can choose r things from a set of n things. For example, $C(4,2)=6$ is read "4 choose 2 is 6" meaning there are 6 ways you can choose 2 things from 4 things.

Important Note: Combinations: Order does *not* matter

Example 6: Game Time How many ways can 10 players choose sides to play five-on-five in a game of basketball?

Solution

As in many counting problems, there is more than one way to do the counting. Perhaps the simplest way here is to determine the number of ways one player can choose his or her four teammates from the 9 other players. In other words, determine the number of subsets of size four taken from a set of size 9, or "nine choose four", which is

$$\binom{9}{4} = \frac{9!}{4!5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126 \text{ ways} .$$

Another approach is to find the number of subsets of size 5 taken from a set of size 10 and divide by 2, getting

$$\frac{1}{2}\binom{10}{5} = \frac{1}{2} \cdot \frac{10!}{5!5!} = \frac{1}{2} \cdot \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 126 \text{ ways.}$$

Note: The determination of the number of ways 10 players can play 5-on-5 is similar to the problem stated at the beginning of the section asking how many ways can 12 people sit at two tables, where one table seats 7 and the other table seats 5. The answer is $C(12,5)$ or equivalently $C(12,7) = 792.$

Example 7: Number of World Series

How many possible ways can two teams play a best-of-seven game World Series[4]? [1] By "ways" we mean we are not distinguishing the two teams. For example either Team *A* or Team *B* can sweep the series in 4 games, but we call this *one* possible outcome of WWWW, not *two* as in AAAA when Team *A* sweeps or BBBB when Team *B* sweeps.

Solution: We count the number of 4,5,6, and 7 game series[5], then add them up to get the total number of series. Since the winner wins 4 games and always the last game, we ask how many ways can the series be played *before* the last game. For example in a 6-game series, we find the number of ways the winner can win 3 games in 5 games, which is "5 choose 3" or $C(5,3) = 10$. For all 4,5,6, and 7 game series, we have the following:

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[4] The World Series is a best-of-7 game series.

[5] If we distinguish the two teams, then the total number of outcomes would be twice this number, or 70. For example. if we counted each 6-game series AABABA and BBABAB as a different outcome rather than just WWLWLW.

Number of 4-game series $= C(3,3) = 1$

Number of 5-game series $= C(4,3) = 4$

Number of 6-game series $= C(5,3) = 10$

Number of 7-game series $= C(6,3) = 20$

Adding these series gives 35 outcomes, which is the total number of best-of-seven game series that can be played (again, twice number or 70 if the two teams are distinguished). It is interesting to note that of all the World Series between 1903 and 2014, 11 series of the 35 have never occurred, one being WLLLWWW when the winner wins games 1,5,6, and 7. The most common outcome is the four-game sweep WWWW, which has occurred 17 times.

Naming Results in Mathematics It is a sorry state of affairs in mathematics, the most precise of all disciplines, that it is lax about giving credit to those who made major discoveries. Pascal's triangle was well known by many mathematicians centuries before Pascal described it. So why is it called "Pascal's triangle?" It probably goes back to some person giving credit to Pascal, and others picking up on that. After a while, it becomes "Pascal's triangle." Mathematics is rife with all sorts of equations and theorems attributed to one person which were discovered by someone else.

Example 8: Going to the Movies Three boys and two girls are going to the movies.

a) How many ways can they sit next to each other if no boy sits next to another boy.

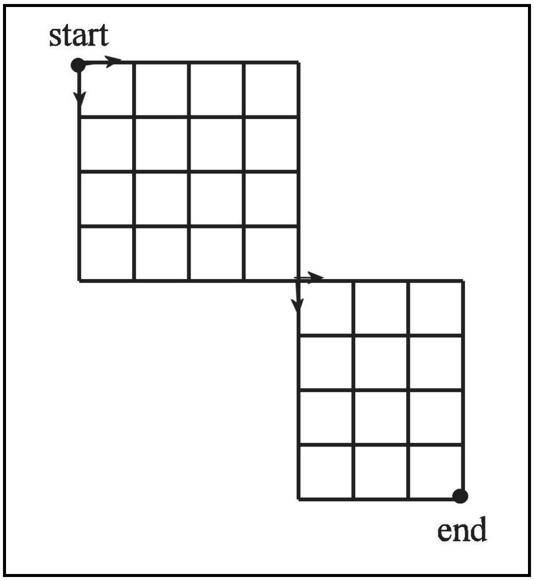
b) How many ways can they sit next to each other if the two girls sit next to each other.

Solution

a) The only way they can sit is boy-girl-boy-girl-boy. But the boys can be permuted $3! = 6$ ways and the girls $2! = 2$ ways, so the total number of arrangements is $3!2! = 12$.

b) Think of the two girls as "one girl" so there are 4 persons; 3 boys and 1 girl. There are $4! = 24$ permutations of these four "individuals". But, for each of these permutations, there are $2! = 2$ permutations of the girls. Hence the total number of arrangements is $4!2! = 48$. ∎

Example 9 How Many Ways Home? How many ways can Mary go home from start to end, in the road system in Figure 4, always moving to the right or down?



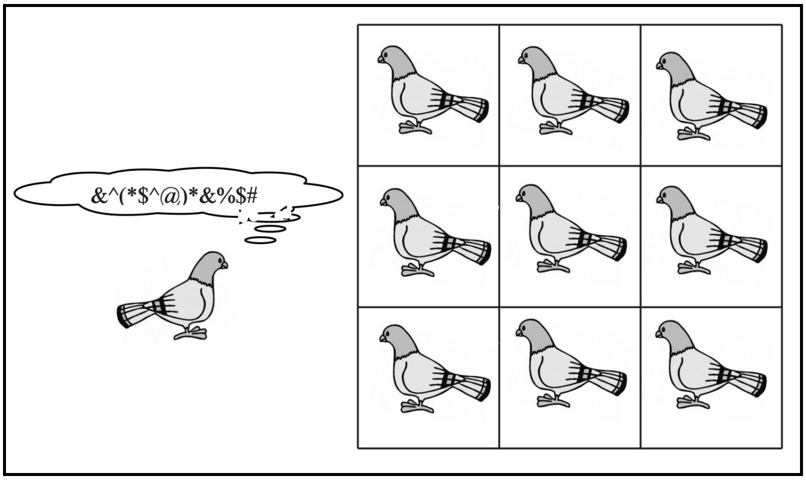
Counting paths
Figure 4

Solution Since all paths pass through the "one-point" gap, the problem is subdivided into two parts; finding the paths from start to the gap, then finding the paths from the gap to end, then multiplying the results together. From start to the gap, note that we travel a total of eight "blocks," four blocks to the right and four blocks down. Labeling each block as R or D , depending whether the move to the right or down, all paths can be written $\{x,x,x,x,x,x,x,x\}$, where four of the $x's$ are R and four are D. Hence, the total number of paths is the number of ways you can select 4 D's (or 4 $R's$) from a set of size 8, which is "8 choose 4" or $C(8,4)=70$. Similarly, the number of paths from the "gap" to end is $C(7,3)=35$ Hence, the total number is

$$\binom{8}{4}\binom{7}{3}=70\cdot 35=2,450 \quad \text{paths..}$$

The Pigeonhole Principle

The pigeonhole principle (or Dirichlet Princple) is based on the observation that if n items are placed in m containers, where $n>m$, then at least one container will contain more than one item. Although the principle seems almost too trivial to yield useful ideas, nothing is further from the truth. Its applications are far reaching and deep and widely used in many fields, such as computer science, mathematical analysis, probability, number theory, geometry, and statistics..



Important Note: People often misinterpret the difference between axioms and definitions. An axiom is an assumed truth whereas a definition assigns names and symbols to given concepts to make it easier to talk about things..

Example 9: Pigeonhole Principle at Work Given any set A of n natural numbers, there will always be two numbers in the set whose difference is divisible by $n-1$.

Solution: When any number in the set is divided by $n-1$, its remainder will be one of the $n-1$ values 0,1,2,..., $n-2$. But the set A has n members so by the pigeonhole principle at least two members of A have the same remainder, say r, when divided by $n-1$. Letting N_1, N_2 be two numbers with similar remainders, we can write

$$\frac{N_1}{n-1} = Q_1 + \frac{r}{n-1}$$
$$\frac{N_2}{n-1} = Q_2 + \frac{r}{n-1}$$

and subtracting gives

$$\frac{N_1 - N_2}{n-1} = Q_1 - Q_2$$

which means their difference is divisible by $n-1$. ∎

Does one take the pigeonhole principle as an axiom or should it be proven from more fundamental principles? The word "obvious" is a loaded word in mathematics, since "obvious" claims of the past have sometimes turned out not only "non-obvious" but not-true[6]. Although the pigeonhole principle seems obvious it can be proven from more basic principles of set theory. Interested readers can find references on the internet.

Example 10: Pigeonhole Principle Goes to a Party There are 20 people at a dinner party, where each person shakes hands with someone at least once. Show there are at least two people who shook hands the same number of times[7].

Solution
Label each person with the number of handshakes they took part in. Each person then has a label of 1 through 19. But there are 20 people at the party, so by the pigeonhole principle, at least two persons must have the same label ▌

Example 11: Points in a Square: Show that for any five points in a square whose sides have length 1, there will always be two points whose distance is less than or equal to $\sqrt{2}/2$.

Solution :Figure 5 shows five points randomly placed in a unit square,. If the square is subdivided into four equal sub-squares, the pigeonhole principle says one of the sub-squares (pigeonholes) contains at least two numbers (pigeons). But the diameter of each sub-square has length $\sqrt{2}/2$, hence the distance between any two points in the sub-square is less than or equal to $\sqrt{2}/2$. ▌

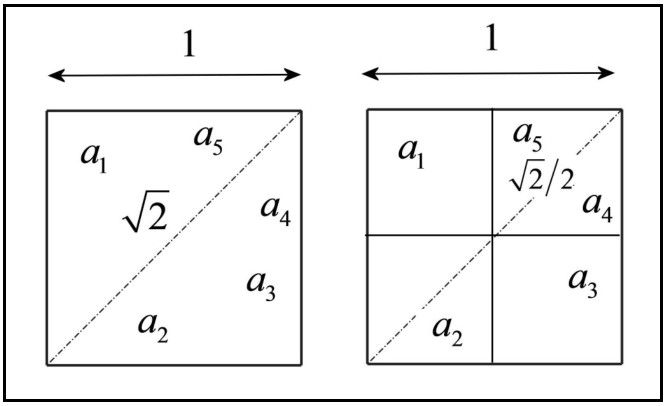


Figure 5

Important Note: The popular example of the pigeonhole is the claim that there are at least two non-bald persons in New York City with exactly the

[6] Many geometric ideas that were taken as fact at the beginning of the 19th century, were overturned by the "arithmetization" of mathematics prevalent in the 19th century.
[7] We assume that people don't shake hand with themselves.

same number of hairs on their head. The argument being no one has more than a million hairs on their head, but there are over a million people living in New York, so by the pigeonhole principle at least two people have the same number of hairs.

Historical Note: The first statement of the pigeonhole principle was made in 1834 by the German mathematician Peter Gustav Lejeune Dirichlet (1805-1859) who called it Sc*hubfachprinzip* ("the drawer principle")..

Problems

1. Compute the following.

a) $P(5,3)$ Ans: $5 \cdot 4 \cdot 3 = 60$
b) $P(4,1)$ Ans: 4
c) $P(30,2)$ Ans: $30 \cdot 29 = 870$
d) $C(4,1)$
e) $C(10,8)$
f) $\binom{7}{2}$
g) $\binom{9}{2}$
h) $(a+b)^6$

2. **Distinguishable Permutations** To find the number of distinguishable permutations of a word like TOOTS, we imagine all five letters are distinguishable from each other, getting 5! permutations. But since we cannot distinguish the 2 Ts and the 2 Os, we divide 2!2!. Find the number of distinguishable permutations

$$\frac{5!}{2!2!} = 30 \ .$$

Find the number of distinguishable permutations of the following letters.

a) TO Ans: $2! = 2$

b) TWO Ans: $3! = 6$

c) TOO Ans: $\frac{3!}{2!}=3$

d) BOOT
e) SNOOT
f) DALLAS
g) TENNESSEE
h) ILLINOIS

3. **Going to the Movies** Find the numbers of ways in which 4 boys and 4 girls can be seated in a row of 8 seats if they sit alternately boy and girl, and if there is a boy named Jerry and a girl named Susan who cannot sit next to each other.

4. **Movies Again** Four couples go to the movies and sit together in 8 seats. How many ways can these people arrange themselves so each couple sits next to each other?

5. **More Movies** Mary and four friends go to the movies. How many ways can they sit next to each other with Mary always between two people?

6. **Baseball Season** A baseball league consists of 6 teams. How many games will be played over the course of a year if each team plays every other team exactly 5 times?

7**. Distinct 3-digit numbers** Determine the number of integers from 100 to 999 that have 2 even digits and 1 odd digit. The even digits are 0,2,4,6, 8 and the odd digits are 1,3,5,7,9.

8. **Interesting Problem** How many 3-digit numbers $d_1d_2d_3$ are there whose digits add up to 8? Note that 063 is not a 3-digit number, it's the 2-digit number 63.

9. **World Series Time** Two teams are playing the best of a 5-game series.

a) What is the total number of series that can be played? One series is WWW, which denotes a 3-game sweep, another is WLWW, which is the result of the winning team winning games 1,3, and 4.

b) What is the total number of series that can be played in a $2n+1$ game series? Note: $n=3$ corresponds to the World Series and we have seen the total number of World Series is 35.

10. **Bell Numbers** A partition of a set A is a collection of nonempty, pair-wise disjoint subsets of A whose union is A. The number of partitions of a set of size n is called the Bell number B_n of the set. For example, when $n=3$ the Bell number is $B_3=5$ meaning a set of size 3 has 5 partitions. The 5 partitions of the set $A=\{a,b,c\}$ are:

$$\{\{a\},\{b\},\{c\}\}$$
$$\{\{a\},\{b,c\}\}$$
$$\{\{b\},\{a,c\}\}$$
$$\{\{c\},\{a,b\}\}$$
$$\{\{a,b,c\}\}$$

Enumerate the partitions of the set $\{a,b,c,d\}$ to find the Bell number B_4.

11. **Two Committees** What is the total number of ways the Snail Darter Society, which consists of 25 members, elect an executive committee of 2 members and an entertainment committee of 4 members, if no member can serve on both committees?

12. **Serving on More than One Committee** How many different ways can the Snail Darter Society, which has 25 members, elect an executive committee of 2 members, an entertainment committee of 3 members, and a welcoming committee of 2 members, if members can serve on more than one committee?

13. **Counting Softball Teams** A college softball team is taking 25 players on a road trip. The squad consists of 3 catchers, 6 pitchers, 8 infielders, and 8 outfielders. Assuming each player can only play her position, how many different teams can the coach put on the field?

14 **Single-Elimination Tournament** Suppose we have a 64-team, single elimination tournament. What is the total number of possible outcomes of the tournament? What is the total number of games played in the tournament?

15. **Permutations as Groups** The $3!=6$ permutations of set $\{1,2,3\}$ can be illustrated by the following 2×3 arrays:

$$\alpha=\begin{bmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{bmatrix} \quad \beta=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{bmatrix}$$

$$\gamma = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \quad \delta = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\varepsilon = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \quad \eta = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

where the bottom row shows how the top row is permuted. For example the array β means position 1 moves to position 2, position 2 moves to position 3, and position 3 moves to position 1. Carrying out one permutation followed by another defines a multiplication of the permutations. For instance, $\varepsilon = \delta\beta$ means we do permutation δ first and β second. Compute the following.

a) $\alpha\beta$
b) α^2
c) β^2
d) $\delta\varepsilon$

16. **The Josephus Problem** A thousand Roman slaves are put in a circle numbered from 1 to 1000, waiting to be executed. The executions begin with slave #1 and proceeds around the circle, where after each execution, the executed slave is removed from the circle and the next slave is skipped over. If there are 6 slaves, the order the slaves are executed is 1,3,5,2,6 with slave 4 last. Suppose the executioner agrees to free the last standing slave. Calling $p(n)$ the position of the last surviving slave when there are n slaves in the initial circle, determine the following.

a) $p(n)$ for each of $n = 1,2,...,10$ slaves.
b) If n is odd, show $p(n)$ satisfies the recurrence relation

$$p(2n+1) = 2p(n)+1, \ p(1) = 1.$$

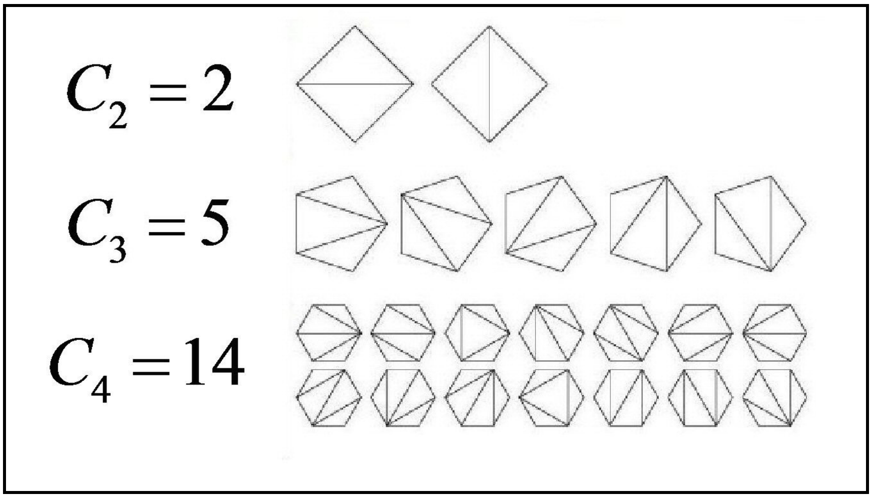
c) If n is even, show $p(n)$ satisfies the recurrence relation

$$p(2n) = 2p(n)-1, \ p(2) = 1.$$

d) Which position would you choose if there are 100 slaves in the circle?

17. **Catalan Numbers** The Catalan number $C_n, n = 1,2,3,...$ is the number of different ways a convex polygon with $n+2$ sides can be subdivided into

triangles. Figure 6 shows the Catalan numbers C_2, C_3, and C_4. Can you find the Catalan number C_5?



First 4 Catalan numbers
Figure 6

18. **Famous Apple Problem** We wish to distribute 8 identical apples to 4 children. How many ways is this possible if each child gets at least one apple?

19. **Derangements** A derangement is a permutation in which none of the elements remain in their natural order. For example, the only derangements of $(1,2,3)$ are $(3,1,2)$ and $(2,3,1)$. Hence, we write $!3 = 2$. Nicolas Bernoulli proved that the number of derangements of a set of size n is

$$!n = n!\sum_{k=1}^{n}\frac{(-1)^k}{k!}$$

Find the number of derangements for $(1,2,3,4)$. Enumerate them.

20. **Counting Handshakes** There are 20 people in a room and each person shakes hands with everyone else. What is the total number of handshakes?

21. **Counting Functions** How many functions are there from $A = \{a,b,c\}$ to $B = \{0,1,2\}$? Write them down and draw the graphs for a few of them.

22.**Combinatorial Euclidean Algorithm** Finding the greatest integer that divides two positive integers can be interpreted as a combinatorial problem where one seeks all possibilities of divisors. The method of finding the greatest common divisor, denoted $\gcd(n,m)$, of two natural numbers n,m is called the **Euclidean Algorithm** and avoids the combinatorial headache of looking through all divisors. The algorithm makes the fundamental observation that the quotient of n divided by m can always be written

$$\frac{n}{m} = q + \frac{r}{m}\ ,\ 0 \le r < m$$

where q is the quotient of n/m and r is the remainder. Using the basic property[8] $\gcd(n,m) = \gcd(m,r)$, it is possible to find a sequence of "decreasing" gcd's, which eventually lead to an obvious gcd. For example, the greatest common divisor of 255 and 68 is 17 as seen from the "decreasing" sequence of gcds:

$$\gcd(255,68) = \gcd(68,51) = \gcd(51,17) = \gcd(17,0) = 17$$

The numbers 51, 17, 0 in the gcds are the remainders of

$$\begin{aligned} &\text{remainder of } \frac{255}{68} \text{ is } 51 \\ &\text{remainder of } \frac{68}{51} \text{ is } 17 \\ &\text{remainder of } \frac{51}{17} \text{ is } 0 \end{aligned}$$

Use the Euclidean Algorithm to find the greatest common divisors of the following pairs of numbers[9].

a) $\gcd(25,3)$
b) $\gcd(54,36)$
c) $\gcd(900,22)$
d) $\gcd(816,438)$

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[8] This is true since any whole number that divides both m and n also divides n and $r = n - mq$, and vice versa.

[9] There are other methods for finding the greatest common divisor of two natural numbers. One could factor each natural number into its prime components, find the common factors, then take the product of the common factors. For example, 816 and 438 have factors of 2 and 3, and so 6 would be their greatest common divisor.

hs23. **Partitions** A partition of a natural number n is a sequence of natural numbers $a_1 \ge a_2 \ge \cdots$ such that $n = a_1 + a_2 + \cdots + a_k$ for some k. There are two partitions of 2, which are $2 = 2,\ 2 = 1+1$, which we denote by 2, 11. There are three partitions of 3, which are $3 = 3,\ 3 = 2+1,\ 3 = 1+1+1$, which we denote by 3, 21, 111. Find the partitions of 4, 5 and 6. You are not asked to find the partitions of $n = 200$ since there are 3,972,999,029,388 of them.

24**. Euler's Theorem** There are many identities related to integer partitions. The first one was due to the Swiss mathematician Leonhard Euler who proved that the number of partitions of a number that contain only odd numbers (1,2,.., 9) is the same as the number of partitions where all the numbers in the partition are different. For example, the number 5 has a total of 7 partitions shown below. Note that there are 3 partitions that only contain odd numbers, and 3 that contain distinct numbers.

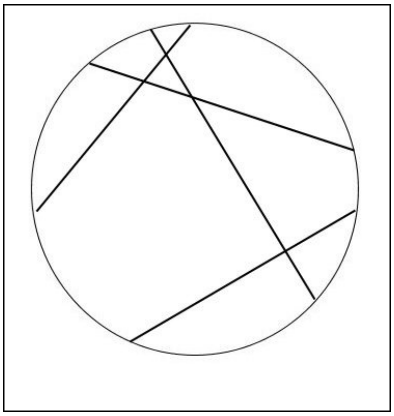
- $5 = 5$ $\leftarrow$ odd numbers only and distinct numbers
- $5 = 4+1$ $\leftarrow$ distinct numbers
- $5 = 3+2$ $\leftarrow$ distinct numbers
- $5 = 3+1+1$ $\leftarrow$ odd numbers only
- $5 = 2+2+1$
- $5 = 2+1+1+1$
- $5 = 1+1+1+1+1$ $\leftarrow$ odd numbers only

Find the partitions of the number 6 and verify Euler's partition theorem.

25. **Fun Problem** How many 3-digit integers are there between 000 and 999 where the middle digit is the average of the other two digits? For example, 246 and 420 satisfy the condition.

26. **Basic Pizza Cutting** You are given a circular pizza and ask to cut it into as many pieces as possible, where a cut means any line that passes all the way through the pizza, although not necessarily through the center. The drawing below shows a pizza with four typical slices dividing the pie into 9 pieces of pizza.[10]

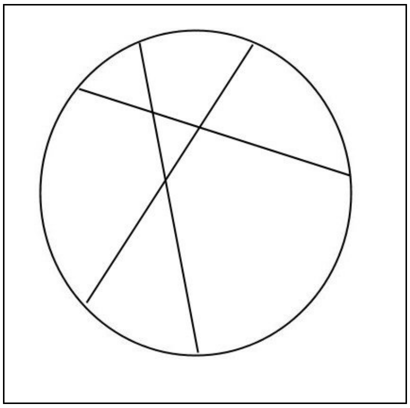
[10] We could just as well say that the entire plane can be subdivided into 9 disjoint sections. In fact, normally the "pizza problem" is stated in terms of the entire plane and not simply the inside of a circle.



Typical pizza with 4 cuts and 9 pieces

a) How many different pieces can you get with three slices?
b) How many different pieces can you get with four slices?
c) Show how you can slice a pizza with 4 slices to obtain 5, 6, 7, 8, 9, 10, and 11 pieces?

27. **Pizza Cutter's Formula** A pizza cutter wants to cut a pizza in such a way that it has the maximum number of pieces, not necessarily the same size or shape. The only restriction on the cut is that it much pass all the way through the pizza, not necessary through the center.



Typical pizza after 3 cuts with 7 pieces

a) If $p(n)$ is the maximum number of pieces from n cuts, where $n = 0,1,2,...$, then $p(n)$ satisfies

$$p(n+1) = p(n) + n + 1, \; p(0) = 1 .$$

b) Show the pizza cutter's formula is

$$P(n) = \frac{n^2 + n + 2}{2} \; , \; n = 0,1,2,... \; .$$

Pigeonhole Problems

To solve problems using the pigeonhole principle, you must find the pigeons and the holes and determine that you have more pigeons than holes.

28. **Pigeonhole at Work** If you pick five numbers from the integers 1 through 8, two of them will add up to 9.

29. **Party Time** At any gathering of two more people, there will always be at least two persons who have the same number of friends.

30. **Drawing Cards** If you pick five cards from a deck of 52 cards, then at least two of them will have the same suit.

31. **Integer Problem** If you pick five numbers from integers 1 to 8, then two of them must add up to nine

32. **Same Grades** Sixteen students in a class are given grades either A,B,C,D, or E. Show that four students must have the same grade.

33. **Mutual Friend or Mutual Strangers**. In any group of six people, there are either three mutual friends or three mutual strangers.

ΓΣΘΨΞΠΩ