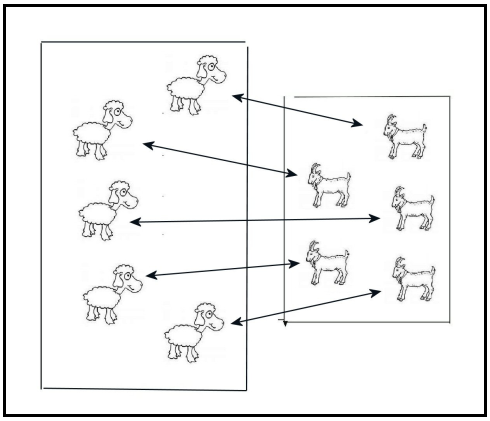
Section 2.4 Cardinality of Sets

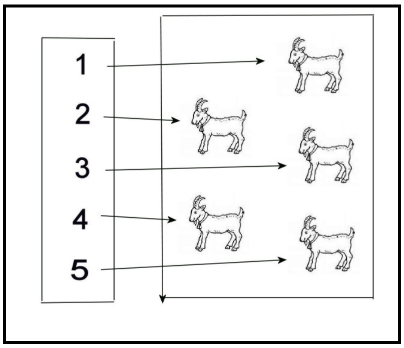
Purpose of Section: To introduce the concept of **cardinality** of a set and **set equivalence** of two sets. We also define what it means for a set to be finite and infinite and how to compare sizes of sets, both finite and infinite. We end our discssion by defining the cardinality of the natural numbers.

Introduction

No one knows exactly when people first started counting, but a good guess might be when people started accumulating possessions. Long before number systems were invented, people might have determined if they had the same number of goats and sheep by simply placing them in a one-to-one correspondence with each other.



Some clever person might even have designated a stone for each goat, obtaining a one-to-one correspondence between the person's goats and a pile of stones. Today we no longer need stones to enumerate things since we have *symbolic* ones in the form of 1, 2, … to represent stones. To determine the number of goats, we simply "count," 1,2, …and envision the rocks $R = \{1, 2, 3, 4, 5\}$ in our mind.



Throughout the history of mathematics, the subject of infinity has been mostly taboo, more apt to be included in a discussion on religion or philosophy. The Greek philosopher Aristotle (*circa* 384-322 b.c.), one of the first mathematicians to think seriously about the subject, said there were two kinds of infinity, the *potential* and *actual*. He said the natural numbers 1, 2, 3, … are *potentially* infinite since they can never be completed and are not a complete entity. The philosopher and theologian Thomas Aquinas (1225-1275) argued that with the exception of God, nothing is actual infinite.

In the 1600s the Italian astronomer Galileo made an observation concerning the perfect squares 1, 4, 9, 16, 25, … Since they constitute a subset of the natural numbers, he argued there should be "fewer" of them than the natural numbers, and Figure 1 would seem to bear this out.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1			4					9							16

More natural numbers than perfect squares
Figure 1

However, Galileo also observed that when one *lines up* the perfect squares as in Figure 2, it appears that both sets have the *same* number of members.

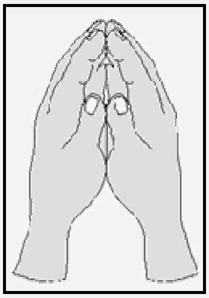
1	2	3	4	5	6	7	8	9	10	11	…	n
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$		$\updownarrow$
1	4	9	16	25	36	49	64	81	100	121	…	n^2

Equal number of perfect squares as natural numbers.
Figure 2

His argument was that for every perfect square n^2, there is exactly one natural number n, and conversely, for every natural number n there is exactly one square n^2. He concluded that "less than," "equal", and "greater than" applied only to finite sets and not infinite ones.

Cantor's Seminal Contribution to Infinity The ground-breaking work of German mathematician, Georg Cantor (1845-1918), whose seminal insights transformed our thinking about "potential" versus "actual" infinities. Many mathematicians resisted Cantor's ideas, but by the time of Cantor's death in 1918, his ideas were accepted for their importance.

Cantor's big discovery was the realization that although it is not possible to count the members of an infinite set, it is possible to determine if infinite sets contain the same number of members, by simply matching members of one set with members of another set. It is similar to the way you determine that you have the same number of fingers on one hand as on the other hand. You simply place the thumb of one hand against the thumb of your other hand, then place your index finger of one hand against your index finger of your other hand, and do the same for your remaining fingers. When you are finished your fingers are matched with each other in a *one-to-one correspondence,* every finger on one hand having a "kindred-soul" on the other hand. You may not know how many fingers you have, but you know both hands have the same number.



The genius of Cantor's theory is that it applies to infinite sets. It doesn't matter that you can't count infinite sets. The above discussion motivates the following formal definitions related to Cantor's fundamental idea.

Cardinality, Equivalence, Finite, Infinite, ...

We begin our trip of sizing up sets by defining several important ideas.

Definitions

- The **cardinality** of a set *A* is the number of members of *A* and denoted by $|A|$. Two sets *A* and *B* have the **same cardinality** or **cardinality number** if their members can be placed in a one-to-one correspondence with each other. Such sets are called **equivalent sets**, which we denote by $A \approx B$. For example, $A = \{a,b,c\}$ and $B = \{1,2,3\}$ are equivalent sets since $a \leftrightarrow 1,\ b \leftrightarrow 2,\ c \leftrightarrow 3$ describes and one-to-one correspondence between the members of *A* and *B*. Thus, we would write $A \approx B$.

- A nonempty set A is **finite** if and only if it is equivalent to a set of the form $\mathbb{N}_n = \{1,2,...,n\}$ for some natural number n. In this case the set has **cardinality** n, and denoted by $|A| = n$. For example, the set $A = \{a,b,c\}$ has cardinality $|A| = 3$ since its members can clearly be put in a one-to-one correspondence with members of the set $\mathbb{N}_3 = \{1,2,3\}$.

• A set is **infinite** if it is not finite. The natural, rational and real numbers are all examples of infinite sets. The members of any of these sets can not be placed in a one-to-one correspondence with members of a set of the form $\{1,2,3,...,n\}$ for some natural number *n*. (There will always be some members of $\mathbb{N},\mathbb{Q}$ and $\mathbb{R}$ left over.)

We now know there are two kinds of sets when it comes to size, finite and infinite. If a set is finite in size, its cardinality can be found simply counting the members of the set. For example, the cardinality of $A=\{a,b,c\}$ is $|A|=3$ and the cardinality of $B=\{a,b,c,d\}$ is $|B|=4$, and so we have $|A|<|B|$. Of course if sets are finite but large, as we have seen in Section 2.3, this idea may be simple conceptually but not in practice.

But what about the cardinality of infinite sets, like $\mathbb{N},\mathbb{Z}$ and $\mathbb{Q}$? What are their cardinalities and do they all have the *same* cardinality and if not, which "infinity"is larger? Is there more than one infinity?

Comparing sizes of finite sets, like the above sets *A* and *B*, poses no problems, but what if finite sets are *large* or what if they are infinite? How do we compare the size of different sets? Fortunately, we have a good friend at our disposal, and a friend the reader is well familiar. It is the concept of the function. Although we formally will not introduce functions until the next Chapter 3, readers will no doubt have an adequate understanding of the basic idea of a function from earlier studies. We begin by introducing three important types of functions $f:A\to B$ that map a domain *A* into a set *B;* the one-to-one function, the onto function, and the one-to-one correspondence function.

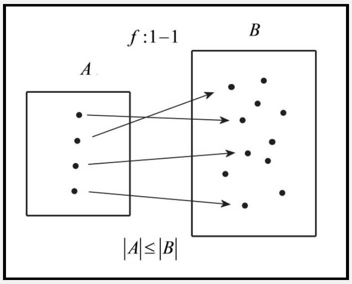
One-to-One Function: If a function $f:A\to B$ sends *different* members $x\in A$ to *different* values $f(x)\in B$, the function f is called **one-to-one** (1-1) or an **injective function.** In the language of predicate logic, we write this as

$$(\forall x_1,x_2\in A)\left[x_1\neq x_2\Rightarrow f(x_1)\neq(x_2)\right]$$

or equivalently, its contrapositive:

$$(\forall x_1,x_2\in A)\left[f(x_1)=f(x_2)\Rightarrow x_1=x_2\right]$$

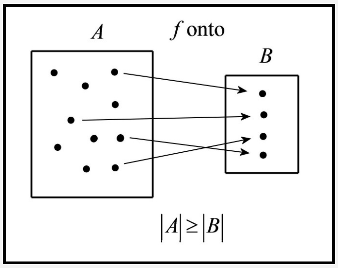
(We might "think" of 1-1 functions as those who "spread out" members of *A* resulting in not "doubling up" on members of *B*. The following figure illustrates a 1-1 function and hints at why 1-1 functions are related to the sizes of *A* and *B*.



The second type of function is the onto function.

Onto Function: A function $f : A \to B$ from A to B is said to be from A **onto** B (or a **surjection**) if every member $y \in B$ is the image of *at least* one pre-image $x \in A$. In other words

$$(\forall y \in B)(\exists x \in A)(y = f(x))$$



We might "think" of onto functions as completely covering up the members of *B,* even if it doesn't require all members of *A* to do it. The following figure illustrates an onto function and hints at why onto functions are related to the sizes of *A* and *B*.

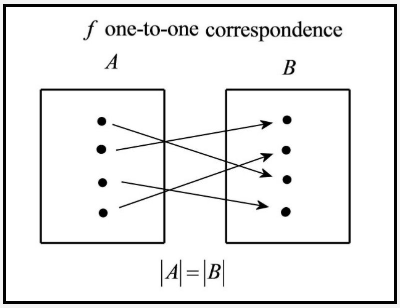
The third type of function we introduce is the one-to-one correspondence.

• **One-to-One Correspondence** A function $f : A \to B$ is called a **one-to-one-correspondence**[1] (or **bijection**) between A and B if it is both one-to-one and onto. We can "think" of a one-to-one correspondence between sets as a relation connecting each member of one set to exactly one member of the other set. The following figure illustrates a one-to-one correspondence

[1] Be careful not to confuse a 1-1 function with a function that is a one-to-one-correspondence. The language is a bit ambiguous.

between A and B and hints at why a one-to-one correspondence is related to the sizes of A and B.

.



Major Result Comparing Sizes of Finite Sets

Theorem 1 (Comparing Finite Sets): Let f be a function that assigns each member of a set A a member of a set B, denoted by $f : A \to B$. We can compare the sizes of A and B using the following rules[2]:

a) If $f : A \to B$ is 1-1, then $|A| \le |B|$.

b) If $f : A \to B$ is 1-1 but not onto, then $|A| < |B|$.

c) If $f : A \to B$ is onto, then $|A| \ge |B|$.

d) e) If $f : A \to B$ is onto but not 1-1, then $|A| > |B|$.

c) If $f : A \to B$ is a one-to-one correspondence, then $|A| = |B|$.

Proof: We prove a) and d) and leave the others as Problems 2,3, 4 in the problem set.

a) The proof is by contradiction, using the logical quivalence

$$(C \Rightarrow D) \equiv [(C \wedge \sim D) \Rightarrow \sim C]$$

In other words, we assume f is 1-1, $|A| \le |B|$, then show f is not 1-1. We begin by using the assumption $|A| \le |B|$ and let

$$A = \{a_1, a_2, ..., a_m\}, \quad B = \{b_1, b_2, ..., b_n\}$$

[2] These "if - then" results can be stated as "if and only if" but we are only interested in the given direction of the theorems.

where $m > n$. Since f is 1-1, we can write[3]

$$f(a_1) = b_1,\ f(a_2) = b_2, \ldots, f(a_n) = b_n,\ f(a_{n+1}) = ?$$

where all the $b_j s$ are distinct. But be have run out of members b_j and so f must map a_{n+1} to one of the previous values of b_j, proving that f is *not* 1-1. Hence, by contradiction the result is proven. ∎

d) We prove

If $f : A \to B$ is onto but not 1-1, then $|A| > |B|$.

by the method of *reductio ad absurdum*

$$(C \Rightarrow D) \equiv \left[(C \wedge \sim D) \Rightarrow \text{logical contradiction} \right]$$

We begin by assuming f is onto, not 1-1, $|A| \le |B|$, then prove a logical contradiction. We let

$$A = \{a_1, a_2, \ldots, a_m\},\quad B = \{b_1, b_2, \ldots, b_n\}$$

where m, n are arbitrary natural numbers. Since f is onto *B*, we know that for each $b_j \in B,\ j = 1,..,n$ there exists an $a_k \in A$ such that $f(a_k) = b_j$. But $|A| \le |B|$, so at least one $a_k s \in A$ maps into two different $b_k \in B$. But this means the function is not a legitimate function (single values map to single values). Hence, by *reductio ad absurdum* the result is proven. ∎

Comparing Infinite Sets

Theorem 1 provides a useful test for comparing the sizes of finite sets even when the sets are large. But this method breaks down when the sets are infinite. However, we can use the results of Theorem 1 to define the meaning of

$$|A| \le |B|,\ |A| < |B|, |A| \ge |B| \text{ and } |A| > |B|.$$

for infinite sets. Since it is intuitively clear that $|A| \le |B|$ when there is a 1-1 function from *A* to *B*, we simply define $|A| \le |B|$ when this is true. Hence we define

[3] There is no reason the specific $a_j s$ have to map to the $b_j s$ with the same index, but there is no reason we can't write the function this way.

Comparing Infinite Sets: We define

$|A| \le |B|$ if and only if there is a 1-1 function from A to B.

$|A| < |B|$ if and only if there is a 1-1 function that is not onto from A to B.

$|A| \ge |B|$ if and only if there is an onto function from A to B.

$|A| > |B|$ if and only if there is an onto function that is not 1-1 from A to B.

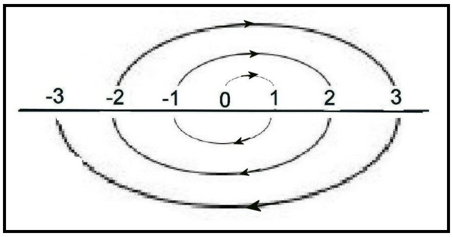
Infinite Sets and Their Cardinalities

We have defined an infinite set to be a set that is not finite. We further define an infinite set A to be **countably infinite** if and only if its members can be arranged in an infinite list $a_1, a_2, a_3, \ldots$. The most obvious example of a countably infinite set being the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, whose cardinality is called **aleph null** and written $\aleph_0$. Thus, we would write $|\mathbb{N}| = \aleph_0$.

Example 1: $\mathbb{Z} \approx \mathbb{N}$ Show that the natural numbers N and the integers $\mathbb{Z}$ have the same cardinality.

Solution

We seek a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$, either as a function or a graphical illustration of a one-to-one correspondence. The diagram in Figure 3 illustrates a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$. Hence, we have $\mathbb{Z} \approx \mathbb{N}$ or $|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$.



One-to-correspondence between $\mathbb{N}$ and $\mathbb{Z}$
Figure 3

A table illustrating a one-to-one correspondence being[4]

$\mathbb{N}$	1	2	3	4	5	6	$\cdots$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$\mathbb{Z}$	0	1	-1	2	-2	3	$\cdots$

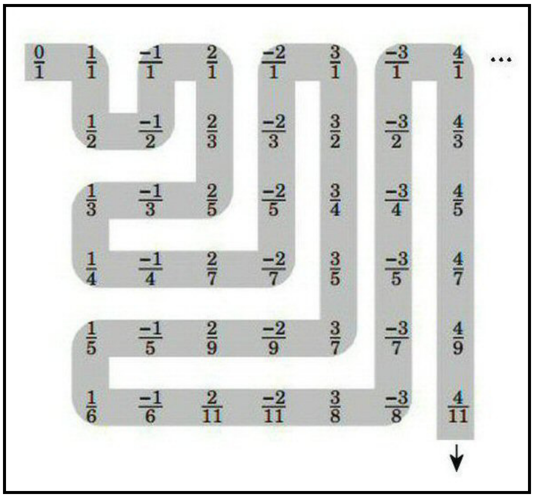
[4] For bijections, we often use double arrows "$\leftrightarrow$" to show the correspondence both ways.

Cardinal and Ordinal Numbers: Numbers are used in two different ways. Numbers can denote both "how many" and "which one." For example, the number 3 is called a **cardinal number** when we say the "three little pigs," but when we say "the third little pig built his house out of bricks,"the number three (or third) is called an **ordinal number**.

Important Note: The symbol "∞", from calculus is not intended to stand for an infinite set. The phrase $x \to \infty$ simply refers to the fact that we allow the variable x to grow without bound.

Theorem 2: Rational Numbers are Countably Infinite
The cardinality of the rational numbers $\mathbb{Q}$ is $\aleph_0$. That is $|\mathbb{Q}| = |\mathbb{N}| = \aleph_0$.

Proof: We show how the rational numbers can be arranged in an infinite list thus showing they are in a one-to-one correspondence with the natural numbers. Such a table is shown in Figure 2 where each column headed with a $0, 1, -1, 2, -2, 3, -3, \ldots$ in the numerator of the given fraction contains all fractions (in reduced form) having that given number as its numerator. (There is no need to add fraction not in reduced form since they would be including more than once.) For example the column headed with $2/1$ contains all the rational numbers with numerator 2. That is $2/1$, $2/3$, $2/5$, $2/7$, $2/9, \ldots$ If you follow the path Figure 2 starting at $0/1 = 0$, you can convince yourself that every rational number is given in this infinite list, demonstrating a one-to-one correspondence between $\mathbb{Q}$ and $\mathbb{N}$. Hence the rational numbers have the same cardinality as $\mathbb{N}$, which we write as $\mathbb{Q} \approx \mathbb{N}$ or $|\mathbb{Q}| = |\mathbb{N}| = \aleph_0$.



Graphical Illustration of $\mathbb{Q} \approx \mathbb{N}$
Figure 2

Interesting Note: We sometimes call 1-1 function an **injective function** and an onto function a **surjective functions**. The words injection and surjection come from the French language, where the French word "injectif" means injecting something into another. The word "sur' in Frnech means "on," hence the word surjection for an onto mapping. The terminology was originally coined by the Bourbaki group, a secret group of French mathematicians named after a French general. Interested readers can learn about this group of mathematicians online.

Cardinality of the Algebraic Numbers:

A real number x is called (real) **algebraic**[5] if it is a real root of a polynomial equation

$$a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 = 0$$

where all the coefficients $a_j, j = 0,1,...,k$ are integers. Rational numbers p/q are examples of algebraic numbers since they are the roots of the linear polynomial equation $qx - p = 0$,. Moreover, the irrational number $\sqrt{2}$ is also algebraic, being a root of $x^2 - 2 = 0$. If a number is not a root of any polynomial equation with integer coefficients, it is called a **transcendental** number. One might think there are more algebraic numbers than transcendental numbers since the first transcendental number was only discovered in 1851. But in 1874, the Cantor shocked the mathematics world when he proved the algebraic numbers are countably infinite. We will see in Section 2.5 just why this fact is so important when we learn about uncountable infinite sets.

Theorem 3 The Set of Algebraic Numbers is Countable The set of algebraic numbers is countably infinite. That is, has cardinality $\aleph_0$.

Sketch of Proof:

Let A_n denote the set of all roots of a polynomial equation of the form

$$a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 = 0$$

where $k \le n$, where, without any loss of generally, we assume the maximum absolute value of the coefficients is less than or equal to n. For example, when $n = 1$ the set A_1 consists of the roots of the three equations $x = 0,\ x - 1 = 0,\ x + 1 = 0$ or $A_1 = \{0,1,-1\}$. The set A_2 consists of the roots of all quadratic and linear polynomial equations where each coefficient in the

[5] Although algebraic numbers can be complex, such as the roots of $x^2 + 1 = 0$, we focus here on real algebraic numbers.

polynomial is one of the values $-2,-1,0,1,2$. A few polynomials of this type are

$$2x^2+x=0$$
$$x^2-x-2=0$$
$$2x+1=0$$
$$x-2=0$$

In general, A_n is the set of real roots of all polynomials of degree less than or equal to n whose coefficients $a_n, a_{n-1},..., a_1, a_0$ can be any one of the values

$$-n,-n+1,...,-2,-1,0,1,1,...,n-1,n$$

The members of the set A_{10} are the real roots of polynomial equations like

$$x^{10}-7x^8+5x^3-10x^2+3x+4=0$$
$$x^4+9x-5=0$$
$$5x+8=0$$

The union of the sets $A_n, n=1,2,...$ defines the algebraic numbers, and since each set A_n is finite, and since a countable union of finite or countable sets is countable, Cantor concluded the algebraic numbers are countable. ▌

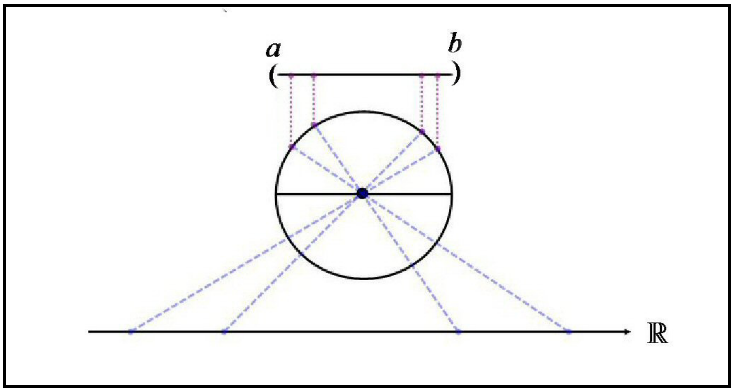
Thus far all of our infinite sets are countably infinite. Are there infinite sets of larger cardinality? Wait until the next section and find out.

Problems

1. Which of the following sets are finite? For finite sets, find if possible, the cardinality of the sets.

   a) any set that can be put in a one-to-one correspondence with a proper subset of itself.
   b) $\{a,b,c,d\}$
   c) all finite strings of the letters a,b. Examples ab, $a, bbaa, aba,...$
   d) the set of rational numbers between 0 and 1.
   e) collection of five-card hands dealt from a deck of 52 cards
   f) prime numbers greater than 10^{10}
   g) $\{n \in \mathbb{N} : n^2 \text{ is odd}\}$
   h) $\{n \in \mathbb{N} : n \text{ is even and prime}\}$

2. **Geometric Proof of Equivalence** Explain how the following drawing provides a geometric proof that any open interval on the real line is equivalent to the real line.



3. **Countable Sets** Show that the union of two countable sets is countable.

4. **Equivalent Sets** For the following intervals, find an explicit one-to-one correspondence that show the intervals are equivalent.

a) $\{a,b,c\} \approx \{1,2,3\}$

b) $[0,1) \approx [0,\infty)$

c) $(0,1) \approx \mathbb{R}$

d) $[0,1] \approx [3,5]$

5. **Cantor-Bernstein Theorem** Given sets A and B, if a subset of A is equivalent to B and a subset of B is equivalent to A, then A and B are equivalent. Use this theorem to prove $(0,1) \approx [0,1]$.

6. **Even and Odd Natural Numbers** Given the set E of even positive integers and O the set of odd positive integers, find a one-to-one correspondence from one set to the other showing the following equivalences.

a) $E \approx O$ Ans: $2n-1 \approx 2n,\ n = 1,2,...$

b) $\mathbb{N} \approx O$

c) $N \approx E$

d) $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$

7. **Infinite Sets** A set is infinite if and only if it is equivalent to a proper subset of itself. Use this property of an infinite set to show the following sets are infinite.

a) $\mathbb{N}$
b) $\mathbb{Z}$
c) $\mathbb{R}$

8. **Bijection from the Prime Numbers** The following sets are equivalent. Find a bijection $f : A \to B$.

a) $A = \mathbb{N}$, $B =$ prime numbers
b) $A = \mathbb{N}$, $B = \{10, 12, 14, ...\}$
c) $A = \mathbb{R}$, $B = (0, \infty)$

9. **Interesting Equivalence** Show $(0,1) \approx (0,1]$

ΓΣΘΨΞΠΩ