Section 2.5 Uncountable Sets

Purpose of Section We present Cantor's seminal result that the real numbers are uncountable, and then go on to present Cantor's proof that the cardinality of the plane is equal to the cardinality of the real numbers,

Introduction

Until now the only infinity considered was the countable infinity of the natural numbers. One wonders if all infinite sets are equivalent to the natural numbers. The great 1874 discovery by Cantor was that there are sets with larger cardinalities than the natural numbers. It was this discovery that led to the development of modern set theory.

Cantor believed, as did all mathematicians at the time, that infinity was infinity and that was all anyone though about it. However, Cantor's attempt to prove that the real numbers had the same cardinality as the natural numbers failed, but it was one of the greatest failures in the history of mathematics since it led to one of the greatest discoveries in mathematics. Namely, that the real numbers have a larger infinity.

Theorem 1 Cantor's Diagonalization Theorem[1] The set of real numbers in the open interval $(0,1)$ is uncountable.

Proof: The 1-1 mapping $f:\mathbb{N}\to(0,1)$, defined by

$$f(1)=\frac{2}{3}$$
$$f(n)=\frac{1}{n}\ ,\ n=2,3,...$$

is 1-1, hence $|\mathbb{N}|\le|(0,1)|$. To show the inequality is a *strict* inequality $(<)$ Cantor assumed the contrary that $|\mathbb{N}|=|(0,1)|$, meaning there exists a one-to-one correspondence between $\mathbb{N}$ and $(0,1)$. We illustrate such a correspondence in Figure 1 where natural numbers are listed on the left and real numbers[2], expressed in decimal form[3], on the right.

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[1] Cantor proved this result, known as his *diagonalization* proof in 1877. He had another proof which he published earlier in 1874.

[2] It is only necessary that we enumerate the real numbers between 0 and 1.

[3] Every real number can be expressed uniquely in decimal form $a_0.a_1a_2a_3...$ where a_0 is an integer and the numbers $a_1,a_2...$ after the decimal are integers between $0\le a_i\le 9$, provided the convention is made that if the decimal expansion ends with an infinite string of 9's , such as 0.499999… which is the same as 0.5, the expansion is modified by raising by 1 the last digit

$\mathbb{N}$				$(0,1)$											
$\downarrow$				$\downarrow$											
1		$\leftrightarrow$		0	.	1	4	3	2	0	2	8	1	4	...
2		$\leftrightarrow$		0	.	3	5	5	4	4	4	6	2	6	...
3		$\leftrightarrow$		0	.	6	3	0	3	5	3	4	1	5	...
4		$\leftrightarrow$		0	.	8	7	8	7	3	5	5	3	3	...
5		$\leftrightarrow$		0	.	0	5	8	6	5	8	8	7	5	...
6		$\leftrightarrow$		0	.	4	9	6	5	8	7	7	5	4	...
..				..		..	..	..	..	..	..	..	..		$\ddots$

Hypothesized one-to-one correspondence $\mathbb{N} \leftrightarrow (0,1)$

Figure 1

Cantor showed that no matter what the one-to-one correspondence there is always a real number not on the list. Hence, there is no mapping from $\mathbb{N}$ onto $\mathbb{R}$, and so $|\mathbb{N}| < |\mathbb{R}|$.

To form Cantor's rogue real number $0.a_1a_2a_3...$ not in the list in Figure 1, Cantor picks the first digit a_1 *different* from the first digit of the real number corresponding to 1 (the number 0.143202814...). In other words, pick a_1 anything different from 1, say $a_1 = 3$. See Figure 2. Now select the second digit a_2 anything different from the second digit of the real number corresponding to the natural number 2 . So we select a_2 different from 5, which we pick $a_2 = 2$.

Continuing this process, working down the diagonal of the table, the first six digits of our rogue number might be 0.327245…and by continuing this process indefinitely, we arrive at a real number that does *not* correspond to any natural number. Hence, the assumed mapping $f : \mathbb{N} \to (0,1)$ is not a one-to-one correspondence since it is not onto. This contradicts the assumption that $\mathbb{N} \approx (0,1)$ so we are forced to conclude the open interval $(0,1)$ has a larger cardinality than the natural numbers. That is $|\mathbb{N}| < |(0,1)|$. ▌

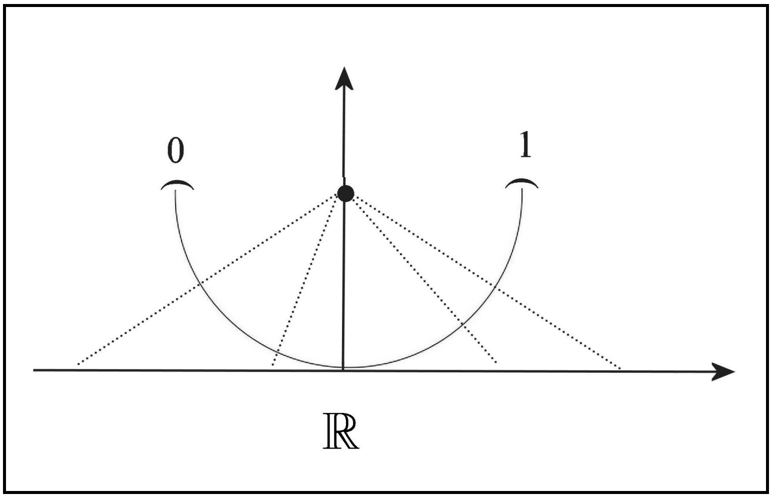
before the 9's and changing all the 9's to 0's. Making this convention provides a one-to-one correspondence between the real numbers and their decimal expansion.

				$\mathbb{N}$						$(0,1)$					
				$\downarrow$						$\downarrow$					
1		$\leftrightarrow$		0	.	[1]	4	3	2	0	2	8	1	4	...
2		$\leftrightarrow$		0	.	3	[5]	5	4	4	4	6	2	6	...
3		$\leftrightarrow$		0	.	6	3	[0]	3	5	3	4	1	5	...
4		$\leftrightarrow$		0	.	8	7	8	[7]	3	5	5	3	3	...
5		$\leftrightarrow$		0	.	0	5	8	6	[5]	8	8	7	5	...
6		$\leftrightarrow$		0	.	4	9	6	5	8	[7]	7	5	4	...
...		...		...		...	...	...	...	...	...	...	..	...	...
...		...		$\downarrow$		$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	...	...	...	...
...		...		0	.	3	2	7	2	4	5	...	...	...	...

Cantor's diagonalization process.
Figure 2

Example 1: Real Numbers are Uncountable Show the open interval $(0,1)=\{x\in\mathbb{R}:0<x<1\}$ has the same cardinality as the real numbers $\mathbb{R}$

Solution We can visualize a correspondence between $(0,1)$ and the real numbers by wrapping the interval $(0,1)$ around the bottom half of a circle whose center lies on the upper y axis of the Cartesian plane. If the real numbers are represented by the x -axis, then the drawing in Figure 3 illustrates the one-to-one correspondence. ▌



Visual proof of $|(0,1)|=|\mathbb{R}|$
Figure 3

Corollary 1: $|\mathbb{N}|<|\mathbb{R}|$

Solution: We saw in Theorem 1 that $|\mathbb{N}|<|(0,1)|$ and Example 1 proved $|(0,1)|=|\mathbb{R}|$. Hence, $|\mathbb{N}|<|\mathbb{R}|$ which states Cantor's seminal result that there is a larger infinity than $\aleph_0$, the cardinality of the real numbers. ▮

Definition: The cardinality of the real numbers is called the **cardinality of the continuum** and denoted by the letter c Sets with cardinality c are called **uncountable sets** (or **uncountably infinite**).

Important Note: Roughly speaking, an uncountable set has so many points its members cannot be arranged in a sequence.

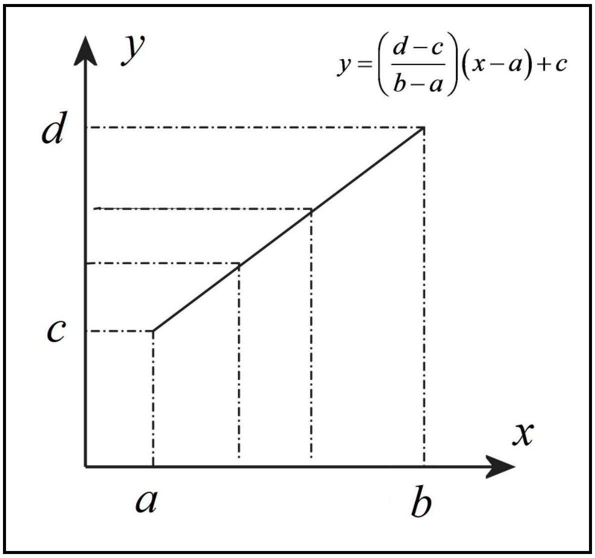
The question we now ask is, what other sets have the cardinality of the continuum. The answer will surprise you.

Example 2: Equivalent Intervals If $[a,b]$ and $[c,d]$ are any two intervals of finite length on the real line, then $[a,b]\approx[c,d]$.

Proof: The function

$$y=\left(\frac{d-c}{b-a}\right)(x-a)+c$$

is a one-to-one correspondence between members of $[a,b]$ and members of $[c,d]$. Figure 4 gives a visual representation of this correspondence.

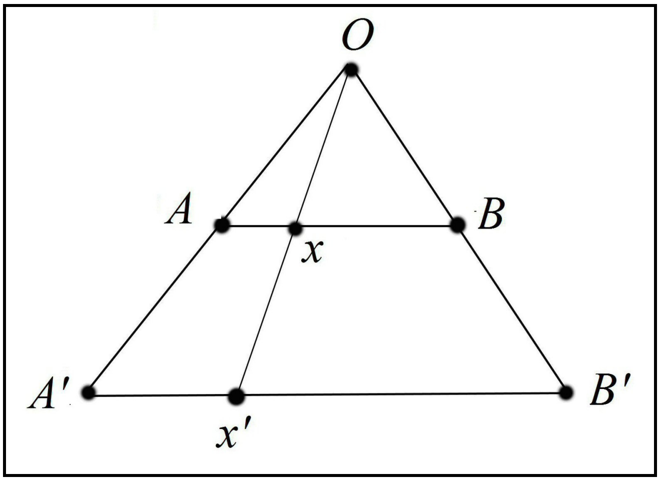


Equivalence of two intervals of real numbers
Figure 4

Important Note: Cantor's proof that there are different "sizes" of infinity is one of the cornerstones of all mathematics, and plays an important role in many areas of pure mathematics such as topology and analysis. In topology, one of the most important concepts is connectedness. One can prove that if a set is connected, it must be uncountable.

Example 3: All Lines are Equal Show that all lines have the cardinality of the continuum.

Solution Figure 5 provides a visual one-to-one correspondence between line segments of different lengths. Any two line segments $A'B'$ and AB of different lengths can be placed in one-to-one correspondence by the bijection $x \leftrightarrow x'$, and combined with the fact that $(0,1) \approx \mathbb{R}$, any interval has cardinality c. Open intervals as large as $(-10^{100}, 10^{100})$, which stretches across many galaxies, and tiny intervals like $(-10^{-100}, 10^{-100})$, whose length is smaller than the width of the smallest atomic particle, all contain the same number of points in Cantor's world.



The shorter segment AB contains as many points as the longer $A'B'$
Figure 5

Important Note: The intervals

$$(a,b),\ [a,b], [a,b),\ (a,b], (-\infty, b,],\ [a,\infty)\ ,$$

all have cardinality c.

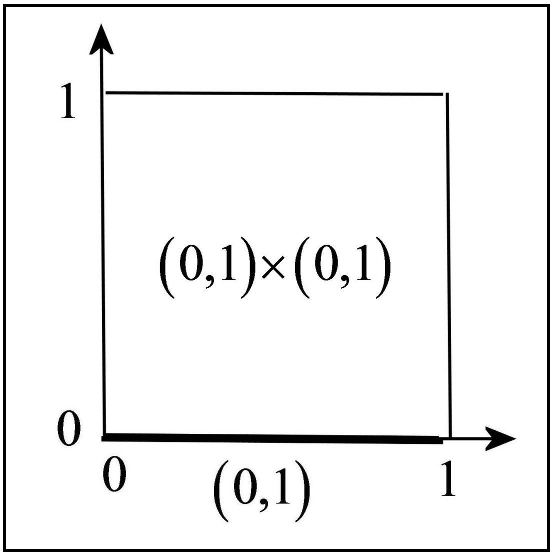
Important Note: The lazy figure eight symbol "∞" does not represent the number infinity; it is simply a symbol used to denote that a set of real numbers is unbounded, such as $(a,\infty), (-\infty,b), (-\infty,\infty)$ and so on.

Cantor’s Surprise

Cantor was not finished surprising the mathematics world with his discovery of a new type of infinity. Cantor's next project was to prove there are more points in the plane than there are real numbers. Again, Cantor failed and again made another important major discovery. Cantor worked tirelessly from 1871 to 1874 to prove the cardinality of the plane is greater than the cardinality of the real line, but to his amazement, he proved they are the same. In a letter to his good friend Richard Dedekind, he said, “I see it, but I don’t believe it.” Here is a summary of Cantor’s proof.

Theorem 2: Cantor's Surprise The cardinality of the open interval $(0,1)$ is the same as the cardinality of the open unit square

$$(0,1)\times(0,1)=\{(x,y)\in\mathbb{R}^2 : 0<x<1,\ 0<y<1\}.$$



$$|(0,1)|=|(0,1)\times(0,1)|$$

Figure 6

Proof: Cantor found 1-1 functions going in both directions between $(0,1)$ and $(0,1)\times(0,1)$. The function $f:(0,1)\to(0,1)\times(0,1)$ defined by

$$f(x)=\left(x,\frac{1}{2}\right),\ \ 0<x<1$$

is a 1-1 function from $(0,1)$ into $(0,1)\times(0,1)$ which shows the inequality $|(0,1)\leq|(0,1)\times(0,1)||$. The function $g:(0,1)\times(0,1)\to(0,1)$ defined by

$$f:(0.a_1a_2a_3...,0.b_1b_2b_3...)\to 0.a_1b_1a_2b_2a_3b_3...\in(0,1)$$

where x, y are written in decimal form [4]

$$x = 0.a_1a_2a_3a_4\cdots$$
$$y = 0.b_1b_2b_3b_4\cdots$$

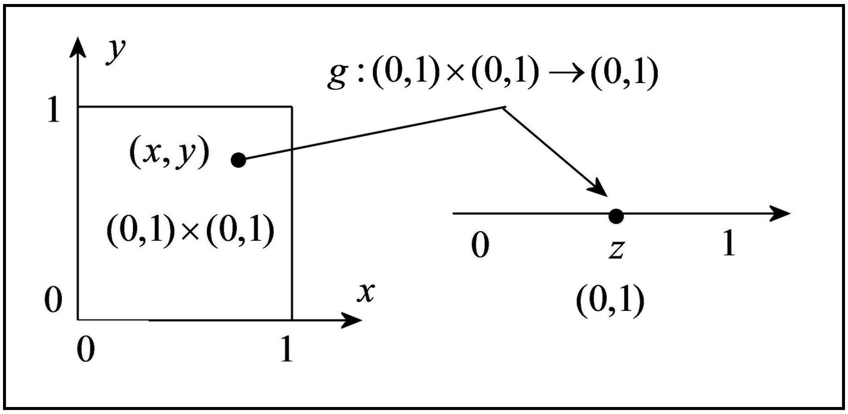
by interlacing the digits of x and y, is 1-1. See Figure 7. From this Cantor concludes

$$|(0,1)\times(0,1)| \le |(0,1)|$$

and combined with the first inequality

$$|(0,1)\times(0,1)| = |(0,1)|$$

and so $(0,1) \approx (0,1)\times(0,1)$.[5]



1-1 map showing $|(0,1)\times(0,1)| \le |(0,1)|$

Figure 7

Summary: In Example 1 we found a bijection $f:(0,1)\to\mathbb{R}$ proving that $|(0,1)| = |\mathbb{R}|$. We now use this function to define a new bijection $F:(0,1)\times(0,1)\to\mathbb{R}$ defined by $F(x,y) = (f(x), f(y))$, proving

$$|\mathbb{R}^2| = |(0,1)\times(0,1)| = |(0,1)| = |\mathbb{R}|$$

In other words, the surprising result that the number of points in the plane is the same as the number of one the real line. In fact, one can continue this line of reasoning to show that the cardinality of n-dimensional space $\mathbb{R}^n$ for any $n = 2,3,\ldots$ is the same as the cardinality of the real numbers.

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[4] To avoid ambiguity, choose 0.5 instead of 0.49999... for the number 1/2. In this way, each decimal form represents exactly one number.

[5] One can also find a 1-1 map from (0,1) to $(0,1)\times(0,1)$ by splitting the digits of a real number into a pair of digits like $a_1a_2a_3a_4a_5a_6\cdots \to (a_1a_3a_5,,,,\ a_2a_4a_6\cdots)$.

Historical Note: Between the years 1871-1884, Cantor created a new and special mathematical discipline, the theory of infinite sets.

Cardinality of the Irrational Numbers: The interval $(0,1)$ is the disjoint union of the rational and irrational numbers. We have seen that the interval $(0,1)$ has cardinality c and that the rational numbers have cardinality $\aleph_0$. Since the union of two countably infinite sets is countably infinite, this implies that the irrational numbers have cardinality c, which is the same as the entire interval. Hence, there are more irrational numbers than rational numbers.

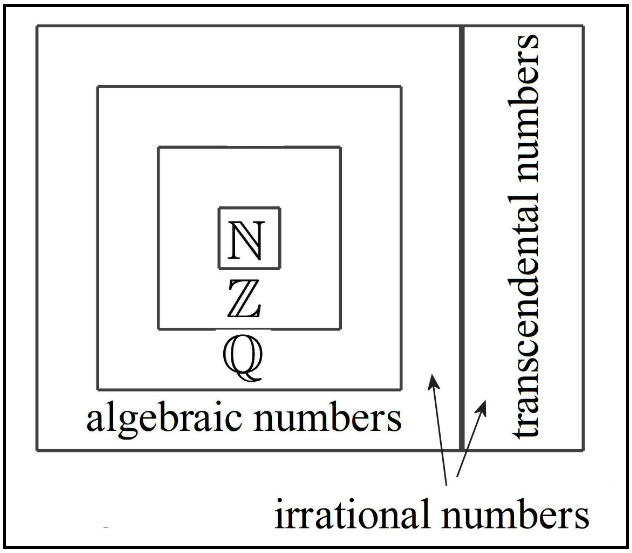
Important Note: One may think the study of infinite sets is simply an academic curiosity for set theorists, but nothing could be further from the truth. Set theory is intimately related to many areas of pure mathematics, notably topology, real analysis, probability, and measure theory. A major concept in topology is the concept of connectedness and connectedness is related to cardinality by a theorem which states only uncountable sets are connected (like the real numbers), and countable sets are disconnected (like the rational numbers). In measure theory, countably infinite sets have measure zero, a concept critical in the study of integration theory.

The Rest of the Story: Transcendental Numbers

We saw in Section 2.4 that the algebraic numbers are countable, and that the real numbers are either algebraic or transcendental. But we have seen in this section that the real numbers have the larger cardinality c .Since the real numbers are the disjoint union of algebraic and transcendental numbers, it implies the transcendental numbers have cardinality c, the same as the real numbers.

There are many important transcendental numbers in mathematics, including π and Euler's constant e. Euler's constant e was proven transcendental by the French mathematician Charles Hermite (1822-1901) in 1873. Nine years later π was proven transcendental by the German mathematician Ferdinand von Lindemann (1852-1939).

Figure 8 gives a schematic diagram of the different number systems within the real numbers.



Number systems within the real numbers
Figure 8

> **Important Note:** The discovery that π is transcendental proved that several of the ancient geometric problems of squaring of the circle, etc., had no solutions.

Problems

1. **Visual Correspondences** Construct a visual one-to-one correspondence between the following sets to show they are equivalent.

   a) two circles of different radii
   b) $(0,\infty)$ and $(-\infty,0)$
   c) $S=(-1,1)$ and $\mathbb{R}$
   d) The points on a sphere minus the top point and points in the plane.

2. **Cardinality of Functions** Show that the cardinality of the set of functions

$$F=\{f:\mathbb{N}\to\mathbb{N}\}$$

is uncountable.

3. **Irrational Numbers** Show the irrational numbers in the interval $[0,1]$ are uncountable.

4. **Bijection from the Unit Square to** $\mathbb{R}^2$ Let

$$S=\{(x,y):0<x<1,\ 0<y<1\}$$

show that the function $F:S\to\mathbb{R}^2$ defined by

$$F(x,y)=(f(x),f(y))$$

where

$$f(x)=\tan\left[\pi\left(x-\frac{1}{2}\right)\right],\ 0<x<1$$

is a bijection from S to $\mathbb{R}^2$. Evaluate the function F at various points inside the unit square S to get a feel for the function.

5. **Visual Equivalence** Let S denote the unit square with vertices $(1,1),(-1,1),\ (-1,-1),(1,-1)$, and let

$$C=\{(x,y):x^2+y^2=1\}$$

denote the unit circle. Show these sets are equivalent with some type of visual drawing.

6. **Algebraic Numbers** Show that the numbers

$$\frac{1+\sqrt{3}}{2} \text{ and } \frac{1-\sqrt{3}}{2}$$

are algebraic numbers.

7. **Transcendental Numbers**

Some transcendental numbers are $\pi, e, e^{\pi}, \pi^{e}, \ln 2, \ldots$ The first demonstrated transcendental number was found by the French mathematician Joseph Liouville in 1844, who constructed and proved the constant

$$L=\sum_{n=1}^{\infty}10^{-n!}$$

is transcendental. Write out the first few decimal digits of this number. Google "Liouville's constant" and read more about it.

8. **Denumerable Plus Denumerable** Prove that if we add another member to a denumerable set A, we still have a denumerable set.

9. **Proving Cardinality** $\mathbb{R} \approx \mathbb{R}^3$ Show that the unit cube

$$C = \{(x, y, z) : x, y, z \in \mathbb{R},\ 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1\} \subseteq \mathbb{R}^3$$

is equivalent to the unit interval $(0,1) \subseteq \mathbb{R}$. Hint: Use the technique Cantor used to prove $(0,1)$ is equivalent to the unit square.

10. **Picking off Members from an Infinite Set** You are given an infinite set of some cardinality $\aleph_0$ or c. If you select an arbitrary sequence of values from the set, there will still be an infinite number left in the original set. Hence, you are left with the observation that a countably infinite set is a subset of any infinite set. What important fact can you conclude from this observation?

ΓΣΘΨΞΠΩ