Section 2.6 Larger Infinities and the ZFC Axioms

Purpose of Section: We state and prove Cantor's (power set) theorem, which guarantees the existence of infinities larger than the cardinality of the continuum. We also state and discuss the Zermelo-Frankel axioms as well as the Axiom of Choice and the Continuum Hypothesis, topics that lie at the center of the foundation of set theory and mathematics.

Cantor's Discovery of Larger Sets

Cantor must have felt he was on a great adventure. He had discovered there were two kinds of infinity, the counting infinity $\aleph_0$ of the natural numbers, and the continuum infinity c of the real numbers. He then wondered if there were other kinds of infinity, infinities even larger than that of c. He also wondered if the infinity of the real numbers was the *next* larger infinity than the infinity of the natural numbers, and would spend the remainder of his life trying to answer that question.

Little ideas often lead to big ideas. We have seen that for finite sets, the power set of a set is larger than the set itself. For example, the set $A = \{a,b,c\}$ contains three elements, whereas its power set

$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

has $2^3 = 8$ elements. This prompted Cantor to ask if the same property held for infinite sets. This question was answered in the affirmative by the following theorem which allows one to create *an infinity of infinites.*

Theorem 1: Cantor's Power Set Theorem The power set $P(A)$ of any set A, finite or infinite, has a cardinality strictly larger than the cardinality of A.

Proof:

We first observe $|A| \le |P(A)|$ since the mapping $f : A \to P(A)$ defined by $f : x \to \{x\}$ is 1.1. To show the inequality is strict, i.e. $|A| < |P(A)|$.Cantor uses his favorite method of proof, proof by contradiction, and assumes $A \approx P(A)$. To make the proof more visual, we let A be the countable set[1] $A = \mathbb{N} = \{1,2,3,...\}$ and assume there is a one-to-one correspondence between the natural numbers and subsets of natural numbers. A typical one-to-one correspondence is shown in Table 1.

[1] We present the proof for countable infinite sets. The proof for uncountable sets follows along similar lines.

$\mathbb{N}$	$\leftrightarrow$	$P(\mathbb{N})$	**Matched or unmatched**
1	$\leftrightarrow$	$\{3,5\}$	**unmatched**
2	$\leftrightarrow$	$\{1,2,7\}$	matched
3	$\leftrightarrow$	$\{9,13,20\}$	**unmatched**
4	$\leftrightarrow$	$\{1,5,6\}$	**unmatched**
5	$\leftrightarrow$	$\{2,5,11,23\}$	matched
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Pairing of the natural numbers and its power set
Table 1

Here is where the proof gets tricky. Note that some numbers, like 1, 3, 4 are not members of the subset of which they are paired, while 2 and 5 belong to the sets they are paired. We call the numbers 1, 3, 4 *unmatched* and the numbers 2 and 5 *matched*. We now form the set of unmatched numbers:

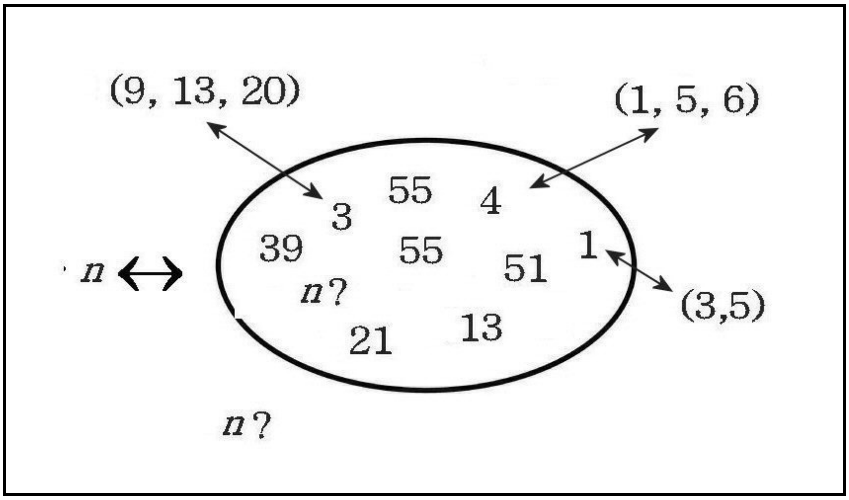
$$\text{unmatched set} = \{n \in \mathbb{N} : n \text{ unmatched}\} \in P(\mathbb{N})$$

But the *unmatched set* is a subset of natural numbers itself, so it must be paired with some natural number, say $n \leftrightarrow$ *unmatched set* as illustrated in Table 2.

$\mathbb{N}$	$\leftrightarrow$	$P(\mathbb{N})$
1	$\leftrightarrow$	$\{3,5\}$
2	$\leftrightarrow$	$\{1,2,7\}$
3	$\leftrightarrow$	$\{9,13,20\}$
$\vdots$	$\vdots$	$\vdots$
n	$\leftrightarrow$	$\textit{unmatched set} = \{1,3,4,\ldots\}$
$\vdots$	$\vdots$	$\vdots$

Does $n \in \text{unmatched set}$?
Table 2

Cantor now asks the fascinating (and tongue-twisting) question, does the number n which is matched with the set of unmatched numbers, belong to the unmatched set? The answer will amaze you.



Does n belong to the set of unmatched numbers?
Figure 1

Yes: If you say n *is* a member of the *unmatched set*, that means n is an unmatched number, but it can't be unmatched if it belongs to the set it is matched. Hence, if n belongs to the unmatched set, it doesn't belong to the unmatched set, i.e. contradiction.

No: If you say n is *not* a member of the *unmatched set,* that means n is a matched number, but n is matched to the *unmatched set* which contradicts the fact that the *unmatched set* only contains unmatched numbers; i.e. a contradiction.

Hence, we have proven

$$n \in \textit{unmatched set} \iff n \notin \textit{unmatched set}$$

which is a contradiction. Hence, we conclude there does not exist a one-to-one correspondence from A to $P(A)$ which means

$$|\mathbb{N}| < |P(\mathbb{N})|.$$ ▌

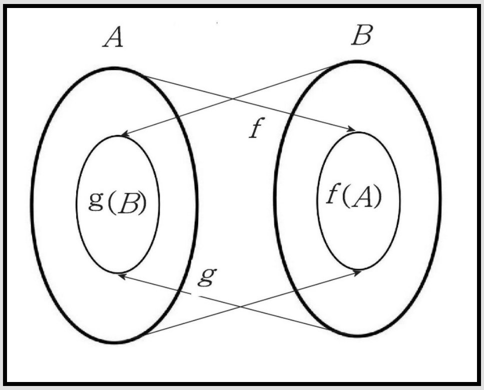
Summary: Since the power set of a set, finite or infinite, has more members than the set itself, it is possible to construct a set of larger infinities by simply taking the power set of a given set. Thus, one can obtain a sequence of infinite sets having larger and larger infinities *ad infinitum*
.

$$\aleph_0 = |\mathbb{N}| < |P(N)| < |P(P(\mathbb{N}))| < |P(P(P(\mathbb{N})))| < \cdots$$

which are all uncountable except for smallest infinity $\aleph_0$. Cantor called all numbers that are not finite **transfinite numbers**, which means numbers larger than finite numbers. Since Cantor's theorem implies that for every set there is a greater set, from which it follows there is *not* a set of all sets. That is, *there is no set of everything*[2] *since there is always more.*

We know that the power set of the natural numbers is a set with larger cardinality than the natural numbers. The question then arises, how does the cardinality of the power set of natural numbers compare with the cardinality of the real numbers? We will prove the important result that $|P(\mathbb{N})| = |\mathbb{R}|$, but first we need the help of the Cantor-Bernstein-Schroeder theorem. The Cantor-Bernstein-Schroeder theorem provides another for showing the equivalence of two sets, and states that if $|A| \le |B|$ and if $|B| \ge |A|$, then $|A| = |B|$. This result sounds obvious when you think about finite sets, but for infinite sets the result is true but requires some serious thought.

Cantor-Bernstein-Schroeder Theorem
Given sets A and B, if there exists a 1-1 mapping $f : A \to B$, and also a $1-1$ mapping $g : B \to A$, then $|A| = |B|$.



Proof: For finite sets A and B the proof is clear since if $f : A \to B$ is 1-1, then $|A| \le |B|$ and if $g : B \to A$ is 1-1, we have $|A| \ge |B|$ and so $|A| = |B|$. For infinite sets, however, this argument does not hole and the proof is quite deep and will not be given here. Interested readers can find proofs online.

The Cantor-Bernstein-Schroeder theorem can be used to show that the open and closed unit intervals (0,1) and [0,1] are equivalent.

[2] The German mathematician David Hilbert said in 1910 that "No one shall drive us from the paradise which Cantor created. Later in 1926, Hilbert said, "It appears to me the most admirable flower of the mathematical intellect and one of the highest achievements of purely rational human activity."

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Corollary of the Cantor-Bernstein-Schoeder

If (0,1) and [0,1] are the open and closed unit intervals on the real line, then $(0,1) \approx [0,1]$

Proof: Let f, g be 1-1 functions defined by

$$f:[0,1] \to (0,1) \quad f(x) = 0.5 + 0.1x$$
$$g:(0,1) \to [0,1] \quad g(x) = 0.5 + 0.1x$$

where f is a 1-1 map from [0,1] into (0,1), and g is a 1-1 map from (0,1) into [0,1]. By the Cantor-Bernstein-Schroeder theorem, we conclude there is a one-to-one correspondence between (0,1) and [0,1] and that $(0,1) \approx [0,1]$, which in turn means they have the same cardinality of the continuum *c*. ▌

We now get to a very important result.

Theorem 2: $P(\mathbb{N}) \approx \mathbb{R}$ and hence $|P(\mathbb{N})| = |\mathbb{R}|$

Proof The goal is to find a one-to-one correspondence between the subsets of $\mathbb{N}$ and members of the interval $[0,1]$. Since we have seen that $[0,1] \approx \mathbb{R}$, this implies that $P(\mathbb{N}) \approx \mathbb{R}$.

We begin by observing that any real number $x \in [0,1]$ can be expressed in binary decimal form $x = 0.b_1 b_2 b_3 \cdots$ where each b_j is 0 or 1. For example

$$x = 0.1101000\cdots = \frac{1}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \cdots = \frac{13}{16}$$
$$x = 0.1111111... = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = 1$$

Note: We must be careful since some numbers in [0,1] have two decimal representations for the same number so we must omit one of them in order to get a one-to-one correspondence. For example, the number 1 can be represented both by 1 and 0.999... and 0.5 can be expressed both by 0.5 and by 0.04999... . In order that each real number in [0,1] has exactly one and only decimal expansion, we use the convention that an infinite string of 9's is never used and replaced by its numeric equivalent (i.e. 0.00999... is replaced by 0.01). Also, infinite strings of 0's are omitted (i.e. 0.101000... is replaced by 0.101). Making these conventions, results in a one-to-one correspondence between [0,1] its binary representations.

We now associate to each real number $x = 0.b_1b_2\cdots$ in [0,1] the subset of $\mathbb{N}$ consisting of indices j for which $b_j = 1$. For example,

$$x = 0.0000...\ 0 \in [0,1] \leftrightarrow \{\ \} = \varnothing \in P(\mathbb{N})$$
$$x = 0.1101\cdots \in [0,1] \leftrightarrow \{1,2,4,...\} \in P(\mathbb{N})$$
$$x = 0.011011\cdots \in [0,1] \leftrightarrow \{2,3,5,6,...\} \in P(\mathbb{N})$$
$$x = 0.11111\cdots \in [0,1] \leftrightarrow \{1,2,3,4,...\} = \mathbb{N} \in P(\mathbb{N})$$

which demonstrates a one-to-one correspondence between the decimal expansions of numbers in [0,1] and subsets of $\mathbb{N}$, which implies $P(\mathbb{N}) \approx [0,1]$. But we have seen that $[0,1] \approx \mathbb{R}$ and so $P(\mathbb{N}) \approx \mathbb{R}$ or $|P(\mathbb{N})| = |\mathbb{R}|$. ▌

The Continuum Hypothesis

If $\aleph_1$ is the next cardinality larger than $\aleph_0$ the question arises as to the relationship between $\aleph_1$ and c. Is $\aleph_1 = c$ or is $\aleph_1 < c$? In other words, is there an infinite set whose cardinality is larger than $\aleph_0$ but less than c? It was Cantor's belief that c was the next larger infinity than $\aleph_0$, i.e. $c = \aleph_1$, but he was never able to prove or disprove it. The hypothesis that c is the next larger infinity after $\aleph_0$ the **continuum hypothesis**:

The Continuum Hypothesis (CH): There is no set S that satisfies $\lvert\mathbb{N}\rvert < \lvert S\rvert < \lvert\mathbb{R}\rvert$

A proof of the continuum hypothesis would confirm that the cardinality of the real numbers c is the smallest uncountable set and bridges the gap between countable and uncountable sets.

After Cantor's death, due to the paradoxes of Bertrand Russell and others, logicians such as Ernst Zermelo and Abraham Fraenkel, set theory was placed on an axiomatic foundation, and in 1938, under the framework of these axioms, the Austrian logician Kurt Gödel (1906-1978) proved that the continuum hypothesis is **consistent** with the axioms of set theory, meaning that adding the continuum hypothesis to existing axioms would not introduce contradictions not already present.

In other words, it meant the continuum hypothesis can not not be proven *false.*[3]. Although this discovery was significant, it was not the final word. The final word came in 1963 when the American logician Paul J. Cohen proved the continuum hypothesis is *independent* of the ZF axioms, the net result being that

[3] In other words, consistent with the Zermelo Fraenkel axioms.

Cantor’s continuum hypothesis $c = \aleph_1$ is **undecidable**. In other words, it is analogous to Euclid’s fifth axiom in that you can accept it as true or accept it as false. If taken as a true axiom along with the other axioms, the resulting theory is called **Cantorian set theory**, while if assumed false, the theory is called **non-Cantorian set theory**). In either case, one has a valid set of axioms, albeit much different. Very strange indeed[4]!

Gödel's Incompleteness Theorem: In 1931, Austrian logician Kurt Gödel (1906-1978) stated and proved, what arguably is, the most famous and far-reaching theorem in the foundation of mathematics. The theorem states that:

Any axiom system containing sufficient axioms to prove elementary arithmetic cannot be both consistent and complete.

In other words, in any consistent axiom system (consistency means one cannot deduce contradictions from the axioms) containing enough axioms to develop basic arithmetic truths, there are true statements that cannot be proven to be either true or false within the axiom system. The implication of this theorem is immense. It says that in a consistent axiom system, there are always statements whose truth value has no meaning. And which statements are those "un-decidable" statements? We will never know.

Need for Axioms in Set Theory

The reader should not entertain the belief that Cantor was the first mathematician to think about sets and their operations, such as union, intersection and compliment. The concept of a set has been known for centuries. Cantor's contribution was to introduce the formal study of *infinite sets* and a deep analysis of the infinite and its many unexpected properties. Although Cantor produced many deep ideas, his interpretation of a set was intuitive. That is, to him a set was simply a collection of objects. It was Bertrand Russell who upset that view of sets with his 1902 discovery, called **Russell’s Paradox**, which showed that the *naïve* view of a set as “any collection of objects” leads to contradictions. Thus, it became clear in order to have a consistent theory of sets (no contradictions), one must formalize the study of sets with “rules-of-the-game.” That is, axioms.

Russell’s Paradox

Russell’s paradox is the most famous of all set-theoretic paradoxes which arises in naïve (non-axiomatic) set theory and motivates the need

[4] Most logicians accept Cantorian set theory. .

for an axiomatic foundation for set theory. The paradox was constructed by English logician Bertrand Russell (1872-1970) in 1901, who proposed a set R consisting of all sets who do *not* contain themselves. He then asked whether the family R was itself a member of R. If $R \in R$, i.e. R *contains* R, then we have a contradiction since R is made up of sets that do *not* contain themselves. On the other hand, if $R \notin R$, this is also a contradiction since R *contains* all sets that do not contain themselves which means $R \in R$. Hence, we are left with the contradictory statement

$$R \in R \Leftrightarrow R \notin R$$

Hence, we must "restrict" the meaning of a set from the naïve point of view which says "a set is an arbitrary collection of elements."

The Zermelo-Fraenkel Axioms

The most commonly accepted axioms of set theory are the Zermelo-Fraenkel[5] axioms (ZFC) axioms, developed by German logicians Ernst Zermelo (1871-1956) and Abraham Fraenkel (1891-1965). Zermelo,[6] who proposed a set of axioms, like those of plane geometry, which restricts the wide latitude of Cantor's interpretation of a set, thus avoiding the "bad" things (i.e. Russell's paradox) but allows enough "objects" to be taken as sets for ordinary use in mathematics. The "C" in "ZFC" refers to the Axiom of Choice, the most controversial and debated of the ten ZFC axioms. Although there is no agreed upon names for each of the 10 axioms, they are often written in varying notation, we have settled on the following list.

Zermelo-Fraenkel Axioms[7]

1.**Axiom of the empty set:** There is a set $\varnothing$ with no elements.

Comment: This set is called the **empty set** and denoted by $\varnothing$. This axiom says that the whole theory defined by these 10 axioms is not vacuous by stating *at least one* set exists.

2. **Axiom of Equality:** Two sets are **equal** if and only if they have the same

[5] There are other axioms of set theory than the ZFC axioms, such as the Von Neumann-Bernays-Gödel axioms (which are logically equivalent to ZFC) and the Morse-Kelly axioms, which are "stronger" than ZF.

[6] Zermelo published his original axioms in 1908 and were modified in 1922 by Fraenkel and Skolem, and thus today are called the Zermelo-Fraenkel (ZF) axioms.

[7] Zermelo's original axioms were stated in the language of second-order logic: i.e. sets were quantified (like $(\forall A), (\exists A)$) in addition to variables. There are versions of the ZFC axioms that use only first-order logic, but for convenience we have quantified sets in a few instances.

members.

Comment: The Axiom of Equality (often called the Axiom of Extension) defines what it means for two sets to be equal. The first two axioms say that the empty set (guaranteed by Axiom 1) is unique.

3. **Axiom of Sets of Size One (singletons exist)** For any a and b there is a set $\{a,b\}$,that contains exactly a and b . That is

$$(\forall a)(\forall b)(\exists A)(\forall z)\left\lceil z\in\{a,b\}\Leftrightarrow(z=x)\vee(z=y)\right\rceil$$

Comment: If we pick $a=b$ then the axiom implies there exists sets with one element $\{a\}$ .Note that the existential quantifier $\exists A$ quantifies a set, which is not allowed in first-order predicate logic.

4. **Axiom of Union of Sets (unions of sets exist)** For any *collection* of sets $F=\{A_\alpha\}_{\alpha\in\Lambda}$, there exists a set

$$Y=\bigcup_{\alpha\in\Lambda}A_\alpha ,$$

called the union of all sets in F . That is,

$$(\forall F)(\exists A)(\forall x)\left\lceil(x\in A)\Leftrightarrow(\exists C\in F)(x\in C)\right\rceil$$

Comment: From Axioms 3 and 4, we can construct finite sets. (We are making progress.)

5. **Power Set Axiom (power sets exist)** For every set A there is a set B (think power set) such that the members of B are subsets of A .

6. **Axiom of Infinity (infinite sets exists)** There exists a set A (think infinite) such that

$$\left\lceil(\varnothing\in A)\wedge(\forall x)(x\in A)\right\rceil\Rightarrow\left\lceil(x\cup\{x\})\in A\right\rceil$$

Comment on the Axiom of Infinity: This axiom "creates" the numbers $A=\{0,1,2,3,...\}$ according to the correspondence: $0=\varnothing$, $1=\{0\}$, $2=\{1\}$, $3=\{2\}$, and so on. We conclude the set of natural numbers exist.

7. **Correct Sets Axiom (axiom that avoids Russell's Paradox)** If $P(x)$ is a predicate and A any set, then

$$\{x\in A:P(x)\}$$

defines a set, i.e. the set of elements for which the predicate $P(x)$ is true. The key point is that the set $\{x\in A:P(x)\}$ is a subset of a previously defined set A, which rules out sets of the form $\{x:x\notin x\}$ which is the Russell paradoxical set. Axiom 7 is what is called an *axiom schema*, which means it is really an infinite number of axioms, one for each predicate $P(x)$.

8. **Image of Sets are Sets** The image of a set under a function is again a set. In other words if A, B are sets and if $f : A \to B$ is a function with domain A and codomain B, then the image

$$f(A) = \{y \in B : y = f(x)\}$$

is a set. (A function can be defined in terms of sets as the set of ordered pairs

$$\{(a,b) \in A \times B : f(a) = b\}$$

9. **Axiom of Regularity (no set can be an element of itself)** Every non-empty set has an element that is disjoint from the set. That is,. for any set

$$(\forall A \neq \varnothing) \Rightarrow (\exists x)\left[(x \in A) \wedge (x \cap A = \varnothing)\right]$$

Comment: As an example if $A = \{a,b,c\}$ then we can pick our element to be "a" where we observe $a \cap \{a,b,c\} = \varnothing$. Note $\{a\} \cap \{a.b.c\} = \{a\} \neq \varnothing$ but $a \cap \{a,b,c\} = \varnothing$. What this axiom accomplishes is to "keep out" sets that contain *themselves*, like $A = \{1, 2, 3, A\}$

10. **Axiom of Choice (AC)** Given *any* collection of non-overlapping, nonempty sets, it is possible to choose one element from each set.

Comments: The axiom of choice is a pure *existence* axiom that claims the existence of a set formed by selecting one element from a collection of non-empty sets. The objection to this axiom lies in the fact that the axiom provides no rule for how the items are selected from the sets, simply that it can be done. Some mathematicians argue that mathematics should not allow such a vague rule.

Comments on the Axiom of Choice

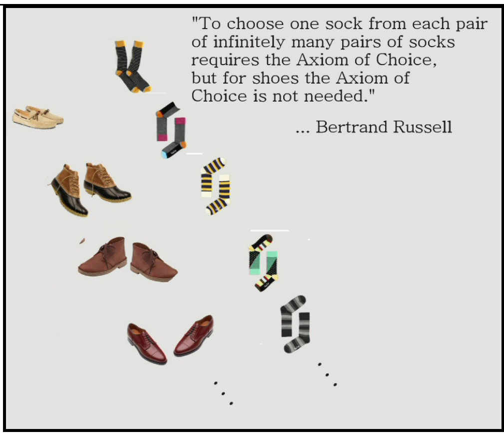
A group of mathematicians are attending a buffet dinner and while they pass through the dessert line containing plates full of different kinds of cookies, one mathematician says she will appeal to the Axiom of Choice and select one cookie from each plate. That's it, that's the Axiom of Choice. The Axiom of Choice says given a family of nonempty sets (plates of cookies), it is possible to select one member from each set. So what's the big fuss over AC? It seems so trivial.

If each plate has a biggest cookie, you can simply pick the biggest cookie from each plate. But what if the cookies look exactly alike, how do you go about selecting a cookie? What is your rule of selection? Can you write it on a piece of paper? Of course not. If you accept the fact that you can select a cookie without specifying a rule, you are appealing to the Axiom of Choice. The Axiom of Choice is a pure *existence* axiom and for that reason it is a bone of contention for some mathematicians who prefer a constructive approach to mathematics. Should the Axiom of Choice be an accepted tool in the mathematician's arsenal??

Bertrand Russell's Shoe Model for AC

Bertrand Russell once gave an intuitive reason why sometimes the Axiom of Choice is required and sometimes it is not. Suppose you are supplied with an infinite number of pairs of shoes and asked to find a "selection" rule for picking one shoe from each pair. In this case, no Axiom of Choice is required, since you can simply choose the left shoe (or right) from each pair.

On the other hand, if you are given an infinite pair of socks and asked the same question. Each sock in a pair looks alike so you are unable to provide a "rule of selection" for choosing a sock from each pair. But the Axiom of Choice comes to the rescue, which says there *is* such a rule, although not explicitly stated, which specifies how to select the socks. It is the non *constructive* aspect of the Axiom of Choice that causes angst to some mathematicians and logicians. To some pure intuitionists, the word "exists" belongs more to religion and not mathematics.



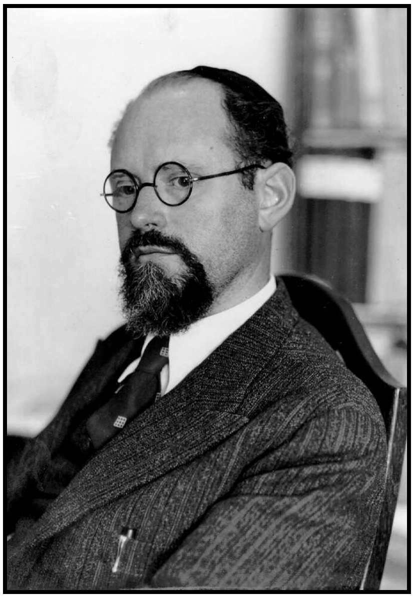
Axiom of Choice ⇔ Well-Ordering Principle

At the time when Zermelo introduced the Axiom of Choice to set theory, most mathematicians accepted it and never gave it a second thought[8]. However, in 1905 Zermelo proved a theorem which gave people second thoughts. We are getting ahead of ourselves, but consider the order relation "$<$" which you have known since middle school, defined on the real numbers $\mathbb{R}$. Now, unlike the interval $[0,\infty)$, the real numbers do *not* have a smallest element. What Zermelo proved was that for *any* (non empty) set, like the real numbers, it is possible to order them in such a way that the set (and any subset) will *always have a smallest element.* This theorem is called Zermelo's **Well-Ordering Theorem** Unfortunately, Zermelo's theorem does not say how to order a set, only that one always. To this day, *no one has ever be any to find an order for the real numbers that has a smallest member.*

What makes Zermelo's Well-Ordering Theorem so perplexing is that on an intuitive level it seems hard to accept, but what is even more perplexing is that Zermelo proved it is equivalent to the Axiom of Choice, which most people feel is obvious.

[8] However, many of the foremost mathematicians of the day did object to the axiom of choice, including measure theory analysts Henri Lebesgue and Emile Borel.

Ernst Zermelo



Abraham Franekel

Conclusion

So what do we know about infinite sets?

Question: Is there more than one type of infinity (or cardinality)?

Answer: Yes, $\aleph_0$ is the smallest infinity but there are larger infinities.

Question: Is there a largest infinity (or cardinality) ?

Answer: There is no largest infinity, Cantor's theorem shows that power sets of sets always yield larger infinities.

Question: But what is the cardinalities of infinites? That is, what is the cardinality of the *set* of all cardinalities?

Answer: Under the ZFC axioms of set theory. Cantor proved there does not exist a set containing all the infinities. Suppose the contrary, that there is a set S containing every cardinal number. Letting T be the union of members of S, and forming the power set $P(T)$, we conclude that since $s \in S$ we have $s \subseteq T$ which in turn implies $|s| < |T| < |P(T)|$, a contradiction. Hence, the question how many infinities are there is meaningless.

Problems

1. **Cardinality of Sets of Functions** Show that the set of all functions defined on the natural numbers with values 0 and 1 has cardinality c. Hint: Relate each sequence of 0's and 1's to a subset of natural numbers and then use Cantor's theorem.

2. **Well-Ordered Sets** A set is said to be well-ordered if every non-empty subset of the set has a least element. The usual ordering $\le$ of the integers is not a well-ordering since the set has smallest element. However, the following relation $\prec$ is a well-ordering of the integers

$$x \prec y \Leftrightarrow \left[\left(|x| < |y| \right) \vee \left(|x| = |y| \wedge \left(x \le y \right) \right) \right]$$

Order the integers according to this well-ordering.

3. **Online Research** Google some of the words and phrases, Zermelo – Fraenkel axioms, Cantor's theorem, Zermelo, well-ordering principle, Cantor's paradox, and learn what others have to say about these topics. Readers of any math book should check the internet for more information.

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