Section 3.3 Equivalence Relations

Purpose of Section To introduce the concept of an **equivalence relation** and show how it partitions a set into disjoint subsets. We also introduce the idea of the congruence of integers and modular arithmetic.

Introduction

The equivalence relation is a relation that holds between two elements that relaxes the sometimes over-restrictive "equals relation" and replaces it by "equals from a certain point of view." This allows us to partition sets into cells where members of a cell, called equivalence classes, share common properties. For example, we might say two lines in the plane are equivalent if they are parallel. It might be useful to consider all parallel lines as one. This motivates the formal definition of an equivalence relation.

Definition An **equivalence relation** on a set A, denoted by "$\sim$", (or sometimes by "$\equiv$") is a relation on A such that for all x, y, z in A, the following properties hold.

Reflexive: $x \sim x$
Symmetric: if $x \sim y$, then $y \sim x$
Transitive: if $x \sim y$ and $y \sim z$, then $x \sim z$.

Example 1: Equivalence Relations Some equivalence relations are the following:

a) $x \sim y$ means $x = y$ for real numbers x, y.
b) $x \sim y$ means x is congruent to y for triangles x, y.
c) $x \sim y$ means $x \Leftrightarrow y$ for logical sentences x, y.
d) $x \sim y$ means "x has the same birthday as y".
e) $x \sim y$ means x differs from y by a multiple of 5
f) $A \sim B$ means sets A, B have the same cardinality.

Example 2: Non-Equivalence Relations
The following relations are *not* equivalence relations.

a) $x \sim y$ means "x is in love with y" on the set of all people.

- Not likely symmetric for one couple.

b) $x \sim y$ means $x \le y$ on the real numbers.

- Not symmetric since $2 \le 3$ does not imply $3 \le 2$.

c) $x \sim y$ means integers x, y have a common factor greater than 1.

- Not transitive since 2 and 6 have a common factor, 6 and 3 have a common factor, but 2 and 3 have no common factors.

d) $x \sim y$ means $x \subseteq y$ on a family of sets.

- Not symmetric since $A \subseteq B$ does not imply $B \subseteq A$.

The Partitioning Property of the Equivalence Relation

We will see that equivalence relations go hand in hand with a partition of a set.

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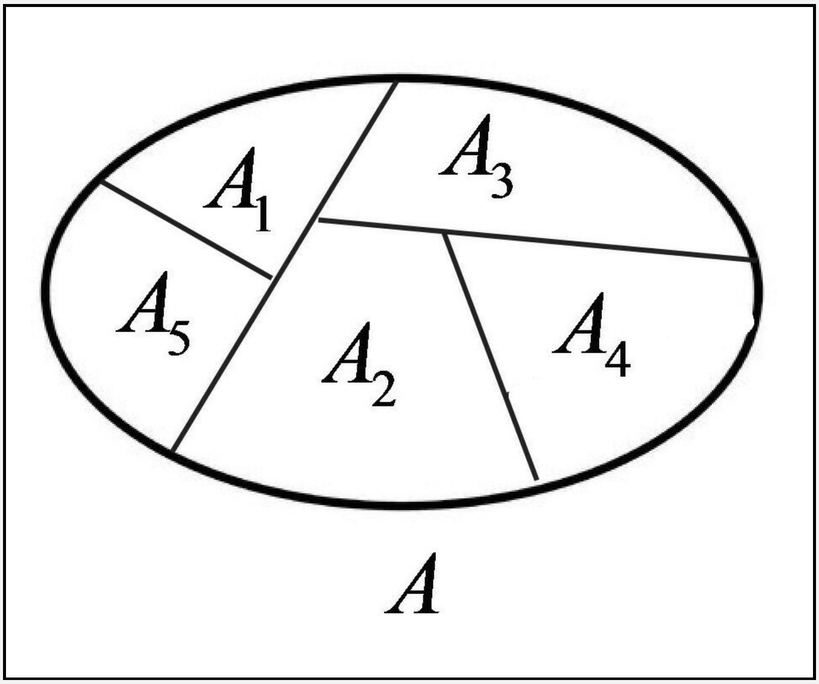
Definition: A **partition** of a set A is a (finite or infinite) collection $\{A_1.A_2....\}$ of nonempty subsets of A, satisfying

- $\bigcup_{k=1}^{\infty} A_k = A$
- $A_i \cap A_j = \varnothing$ for every pair A_i and A_j.



The following theorem reveals the reason equivalences relation play an important role in mathematics...

Theorem 1: Equivalence Classes

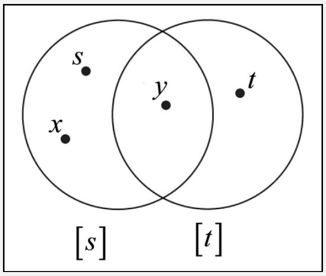
If R is a relation defined on a set A, then

R is an equivalence relation on $A \Leftrightarrow R$ induces a partition of A

Proof: $(\Rightarrow)$ Assume R is an equivalence relation on A and for each $x \in A$, define

$$[x] = \{ y \in A : y \sim x \} \subseteq A$$

which we call the **equivalence class** of x. We now show the set of all equivalence classes in A is a partition of A. Note that every $x \in A$ belongs to some equivalence class, i.e. $[x]$, since every member is equivalent to itself and so the union of the equivalence classes is A. We now show the equivalence classes $[x]$ are disjoint by showing if two equivalence classes intersect each other, they are the same equivalence class.



We now $[s] \cap [t] \neq \varnothing$ and show $[s] = [t]$. If $[s] \cap [t] \neq \varnothing$ there is a $y \in [x] \cap [y]$.

$(\subseteq)$ To prove $[s] \subseteq [t]$ assume $x \in [s]$. Hence

a) $x \sim s$ (since we assumed $x \in [s]$)

b) $y \sim s$ (since $y \in [s] \cap [t] \subseteq [s]$)

c) $s \sim y$ (since $\sim$ is symmetric)

d) $x \sim y$ (since $\sim$ is transitive)

e) $y \sim t$ (since $y \in [s] \cap [t] \subseteq [t]$)

f) $x \sim t$ (since $\sim$ is transitive)

Hence $x \in [t]$ and so $[s] \subseteq [t]$. The proof $[s] \supseteq [t]$ is similar and so $[s] = [t]$.

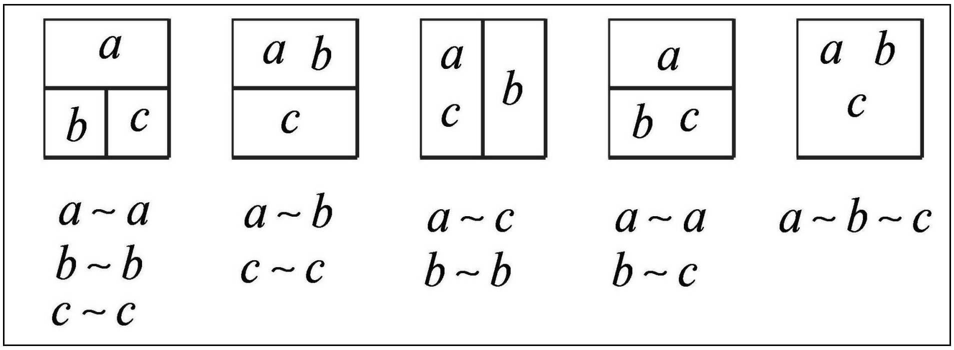
$(\Leftarrow)$ The verification that a partition on a set induces an equivalence relation is left for the reader. See Problem 23.

The number of partitions of a set of size n is called the **Bell number** B_n of the set and its roots go back to medieval Japan. The first few Bell numbers are

$$1,1,2,5,15,52,203,677,4140,21147,115975...$$

The set $\{a,b,c\}$ of three members has a Bell number $B_3 = 5$.

Example 3: Equivalence Relations from Partitions Figure 1 shows the five partitions of the set $\{a.b.c\}$ and the equivalence relations induced by the partitions.



Partitions of $\{a,b,c\}$ and their induced equivalence relations
Figure 1

Modular Arithmetic

Two integers $x, y \in \mathbb{Z}$ are said to be **congruent modulo** ***N***, denoted by

$$x \equiv y \,(\text{mod}\, N)$$

if they have the same remainder when divided by the integer N .If integers x, y have the same remainder when divided by N, then

$$\frac{x}{N} = Q_1 + \frac{r}{N}, \quad \frac{y}{N} = Q_2 + \frac{r}{N}$$

where Q_1, Q_2 are the respective quotients and r the common remainder. Subtracting the two equations gives

$$\frac{x}{N} - \frac{y}{N} = (Q_1 - Q_2) \quad \text{or} \quad x - y = (Q_1 - Q_2)N$$

which implies if x, y are congruent modulo N, then their difference is divisible by N. In other words

$$x \equiv y \,(\text{mod}\, N) \Leftrightarrow (\exists k \in \mathbb{Z})(x - y = kN)$$

We now show that the congruence relation is an equivalence relation.

Theorem 2: Congruence is an Equivalence Relation on $\mathbb{Z}$.

Proof: We show the congruence relation $\equiv$ is reflexive, symmetric, and transitive.

- **reflexive:** $x \equiv x(\text{mod}\, N)$ since N divides $x - x = 0$.
- **symmetric:** If $x \equiv y(\text{mod}\, N)$, then N divides $x - y$. Hence, there exists an integer k such that

$$x - y = kN \text{ or } y - x = -kN = N(-k)$$

which means N divides $y - x$. Hence $y \equiv x(\text{mod}\, N)$ which means $\equiv$ is symmetric.

- **transitive:** For integers x, y, z assume $x \equiv y(\text{mod}\, N)$ and $y \equiv z(\text{mod}\, N)$. Hence,

$$\begin{cases} x \equiv y(\text{mod}\, N) \\ y \equiv z(\text{mod}\, N) \end{cases} \Rightarrow \begin{cases} (\exists k_1 \in \mathbb{Z})(x - y = k_1 N) \\ (\exists k_2 \in \mathbb{Z})(y - z = k_2 N) \end{cases}$$

Adding these equations gives

$$(x - y) + (y - z) = k_1 N + k_2 N$$

or

$$x - z = (k_1 + k_2)N$$

which shows N divides $x - z$ or $x \equiv z(\text{mod}\, N)$. Hence $\equiv$ is a transitive relation.

The congruence relation "$\equiv$" on $\mathbb{Z}$ partitions the integers into **congruence classes** (called **residue classes**), where integers in each residue class have similar remainders when divided by N. For $N = 5$, the residue classes are $[0]_5, [1]_5, [2]_5, [3]_5, [4]_5$ and are listed in the following table.

Residue Classes for $\mathbb{Z}$ Modulo (5)	
$[0]_5 = \{5n : n \in \mathbb{Z}\}$	$= \{\cdots -10, -5, \underline{0}, 5, 10 \cdots\}$
$[1]_5 = \{5n+1 : n \in \mathbb{Z}\}$	$= \{\cdots -9, -4, \underline{1}, 6, 11 \cdots\}$
$[2]_5 = \{5n+2 : n \in \mathbb{Z}\}$	$= \{\cdots -8, -3, \underline{2}, 7, 12 \cdots\}$
$[3]_5 = \{5n+3 : n \in \mathbb{Z}\}$	$= \{\cdots -7, -2, \underline{3}, 8, 13 \cdots\}$
$[4]_5 = \{5n+4 : n \in \mathbb{Z}\}$	$= \{\cdots -6, -1, \underline{4}, 9, 14 \cdots\}$

Note that the residue classes partition the integers into five disjoint sets:

$$\mathbb{Z} = [0]_5 \cup [1]_5 \cup [2]_5 \cup [3]_5 \cup [4]_5$$

The collection of partitions is called the **quotient set** of $\mathbb{Z}$ modulo 5, and denoted by $\mathbb{Z}/5\mathbb{Z}$. In other words

$$\mathbb{Z}/5\mathbb{Z} = \{ [0]_5, [1]_5, [2]_5, [3]_5, [4]_5 \}$$

Modular Arithmetic: Modular arithmetic (also called clock arithmetic) is a system of arithmetic whose numbers "wrap around" after they reach a certain value, called the **modulus**. Modular arithmetic was introduced by Carl Friedrich Gauss at the age of 24 in 1801 in his seminal book on number theory *Disquisitiones Arithmeticae* (Latin: discourse into arithmetic). .

Property	**Reflexive**	**Symmetric**	**Transitive**	**Antisymmetric**
Relation				
$=$	yes	yes	yes	yes
$\leq$	yes	no	yes	yes
$<$	no	no	yes	yes
$\parallel$	yes	yes	yes	no
$\perp$	no	yes	no	no
$\subseteq$	yes	no	yes	yes
$\equiv \bmod (n)$	yes	yes	yes	no
$\cong$	yes	yes	yes	no

Common relations in mathematics

Important Note: Some people do not understand why the remainder of the fraction $-3/5$ is 2. Remainders are non negative integers, so $-3/5 = (-5+2)/5 = -1 + 2/5$.

Example 4: Equivalence Classes in the Plane Cartesian product $\mathbb{N} \times \mathbb{N}$ defines the grid points in the first quadrant of the Cartesian plane (i.e. points with positive integer coordinates). The relation

$$(a,b) \sim (c,d) \Leftrightarrow a + d = b + c.$$

between two points in this set can be shown to be an equivalence relation (See Problem 20). Note[1]

$$(3,1) \sim (4,2) \sim (5,3) \sim \cdots$$

and

$$(1,1) \sim (2,2) \sim (3,3) \sim \cdots$$

This particular equivalence relation gives rise to a very important collection of equivalence classes. What are they?

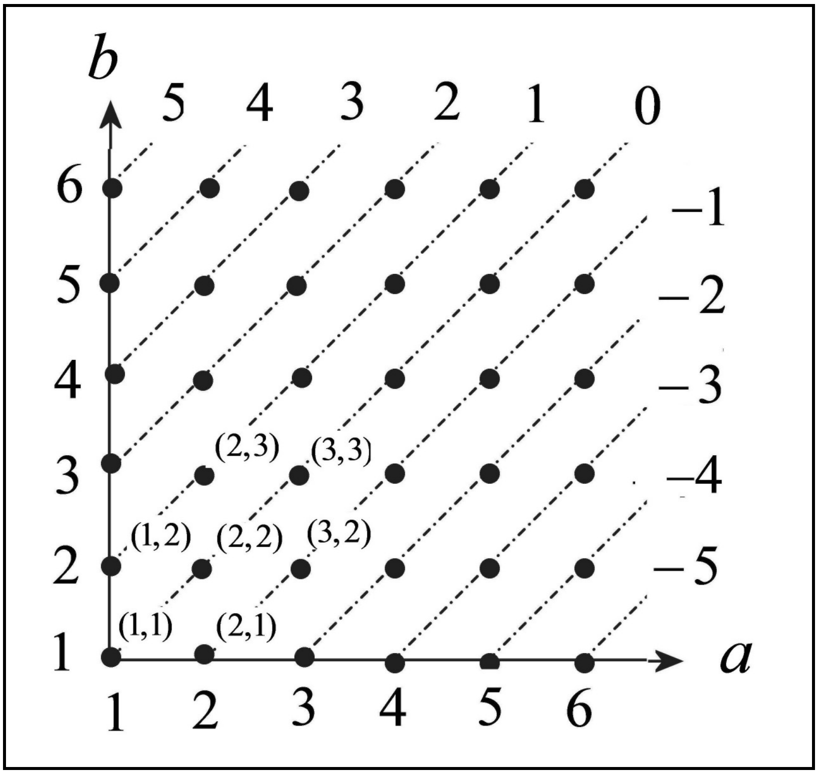
Solution

If two points (a,b) and (c,d) are equivalent when $a-b-c-d$ this means they both lie on a 45-degree line. Hence, the given equivalence relation partitions the grid points in the first quadrant into grid points on disjoint 45-degree lines $y = x+n$, where each value of the integer n represents a different equivalence class. Some typical equivalence classes are

Equivalence Classes	
... ...	
$[(3,1)] = \{(3,1),(4,2),(5,3)...\}$	$(n=-2)$
$[(2,1)] = \{(2,1),(3,2),(4,3)...\}$	$(n=-1)$
$[(1,1)] = \{(1,1),(2,2),(3,3)...\}$	$(n=0)$
$[(1,2)] = \{(1,2),(2,3),(3,4)...\}$	$(n=1)$
$[(1,3)] = \{(1,3),(2,4),(3,5)...\}$	$(n=2)$

These equivalence classes are illustrated in Figure 2 as points with integer coordinates lying on 45-degree lines in the first quadrant.

[1] The reader may wonder why we didn't define the equivalence relation more naturally as $(a,b) \sim (c,d)$ if $a-b=c-d$. The reader will discover the important reason in Chapter 4.



Equivalence classes as grid points on lines $y = x + n$

Figure 2

Problems

1. **Testing Relations** Let A denote the student body at a university and individual students by x and y. Determine if the following relations are equivalence relations on A.

a) x is related to y iff x and y have the same major.
b) x is related to y iff x and y have the same GPA.
c) x is related to y iff x and y are from the same country.
d) x is related to y iff x and y have the same major.

2. **Equivalence Relations** Which of the following relations R are equivalence relations on the given set A. For those relations that are equivalence relations, find the equivalence classes.

a) xRy if and only if $y = x^2$. $(A = \mathbb{R})$
b) mRn if and only if m is a factor of n. $(A = \mathbb{N})$

c) xRy if and only if x and y have the same remainder when divided by 5. $(A=\mathbb{N})$

d) xRy if and only if $|x-y|\le 1$. $(A=\mathbb{R})$

e) $(a,b)R(c,d)$ if and only if $a^2+b^2=c^2+d^2$. $(A=\mathbb{R}^2)$

3. **Not Equivalence Relations** Determine if the following relations are relations, and if not which condition: reflexive, symmetric, or transitive fails?

a) The relation "$\le$" on the real numbers.
b) The empty relation on an empty set (i.e. xRy never true)
c) Relation "$\subset$" of being a proper subset on a family of sets
d) Relation of being perpendicular on lines in the plane.

4. **Finding the Equivalence Relation** Partition $A=\{a,b,c,d,e\}$ into the equivalence classes $\{\{a,c\},\{b,e\},\{d\}\}$. Find the equivalence relation induced by this partition.

5. **Finding the Quotient Set** The relation

$$R=\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,1)\}$$

is an equivalence relation on $A=\{1,2,3,4,5\}$. What is the partition of A induced by this relation?

6. **Finding Equivalence Classes** The set $\{1,2,3,4\}$ is partitioned into $\{\{1,2\},\{3,4\}\}$ by an equivalence relation R. Find the following:

a) $[1]$ Ans: $[1]=\{1,2\}$
b) $[2]$
c) $[3]$
d) $[4]$

7. **Hmmmmmmmmm** If an equivalence relation R on a set A has only one equivalence class, what is the relation?

8. **Equivalence Relation** Let $m,n\in\mathbb{Z}$. We say $m\sim n$ if and only if 3 divides $m+2n$.

a) show $\sim$ is an equivalence relation
b) find the equivalence classes.

9. **Equivalence Relation in Calculus** Given the set of continuous functions $C[0,1]$ defined on the closed interval $[0,1]$, define $R \in C[0,1] \times C[0,1]$ by

$$f \, R \, g \text{ if and only if } \int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx$$

a) Show that R is an equivalence relation.
b) Find $g \in C[0,1]$ equivalent to $f(x) = x$ but $f \neq g$.

10. **Equivalence Relations in Analysis** Let $A = [-1,1]$ and define an equivalence relation R on A by xRy if and only if $x^2 = y^2$, $x, y \in [-1,1]$. Find the equivalence classes.

11. **Equivalence Sets of Polynomials** $P(x)$ are polynomials on the real line and $I \subseteq P(x)$ are polynomials that satisfy $p(0) = 0$. For $f, g \in P(x)$ show that $f \sim g$ defined by $f \sim g \Leftrightarrow f - g \in I$
is an equivalence relation[2].

12. **Modular Arithmetic** If $x, y \in \mathbb{Z}$, we say $x \equiv y \pmod{n}$ if n divides $x - y$ for a positive integer n. Show the relation $\equiv$ is an equivalence relation.

13. **An Old Favorite** The equals relation "$=$" is the most familiar equivalence relation. What are the equivalence classes induced by the equals relation on $A = \{1,2,3,4,5\}$?

14. **Equivalence Classes in Logic** Define an equivalence relation on logical sentences by saying two sentences are equivalent if they have the same truth value. Find the equivalence classes in the following collection of sentences.

a) $1 + 2 = 3$
b) $3 < 5$
c) $2 | 7$
d) $x^2 < 0$ for some real number.
e) $\sin^2 x + \cos^2 x = 1$
f) Georg Cantor was born in 1845.
g) Leopold Kronecker was a big fan of Cantor.
h) Cantor's theorem guarantees larger and larger infinite sets.

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[2] In the language of abstract algebra, the set $P(x)$ is a polynomial ring and the subset I a vanishing ideal in the ring.

15. **Similar Matrices** Two square matrices A, B are equivalent if there is an invertible matrix M such that $MAM^{-1} = B$. Show that similarity of matrices is an equivalence relation.

16. **Counting Equivalence Relations**

a) Count the number of equivalence relations on $A = \{1,2\}$.
b) Count the number of equivalence relations on $A = \{1,2,3\}$.

17. **Arithmetic in Modular Arithmetic** Suppose

$$a \equiv c \pmod 5$$
$$b \equiv d \pmod 5$$

Show

a) $a + b \equiv c + d \pmod 5$
b) $a - b \equiv c - d \pmod 5$
c) $ab \equiv cd \pmod 5$

18. **Mapping into the Equivalence Class** Let X denote student body at your college or university and define the equivalence relation as "being in the same class" (freshman, sophomore, junior or senior). Define the mapping $f : x \to [x]$ that sends each student $x \in X$ into his or her equivalence class $[x]$. Is this a well-defined function? What is your value under this mapping?

19. **Equivalence Classes as Directed Graphs** Inasmuch as equivalence relations are binary relations, they can be represented by digraphs. Draw a digraph that represents the equivalence classes of the set $\{0,1,2,3,4,5,6,7\}$ when two elements are equivalent if they have the same remainder when divided by 3.

20. **Defining Integers from Natural Numbers** Example 4 shows how to define the negative integers and zero in terms of pairs of positive integers by means of an equivalence relation. Show that the relation on

$$(a,b) \sim (c,d) \text{ if and only if } a + d + b + c$$

is an equivalence relation. List the different equivalence classes and observe that the equivalence classes for (a,b) can be associated with the negative integers and zero, thus allowing one to define the negative integers in terms of pairs of positive integers.

21. **Counting Partitions** Find the different partitions of the sets

a) $A = \{1,2\}$
b) $A = \{1,2,3\}$

22 **Interesting Equivalence Relation** Define a relation R on the non-negative integers

$$A = \{0,1,2,3,...,29,30\}$$

by

$mRn \Leftrightarrow$ (product of the digits of m = product of the digits of n).

For example $16R23$, $4R14$.

a) Show that R is an equivalence relation on A.
b) Find the equivalence classes of the relation.
c) The equivalence classes are listed below.

Product	Integers
0	0,10,20,30
1	1,11
2	2,12,21
3	3,13
4	4,14,22
5	5,15
6	6,16,23
7	7,17
8	8,18,24
9	9,19
10	25
12	26
14	27
16	28
18	29

23. **Relations and Partitions** Show that if a relation R on a set A induces a partition of A, then R is an equivalence relation on A

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