Section 5.4 Point-Set Topology on the Real Line

Purpose of Section To introduce some topological concepts of the real number system, such as open and closed sets, interior, boundary, and exterior points of a set, and limit points. These act as the foundation of many concepts in analysis.

Introduction

Point-set topology (sometimes called **general topology**) is the branch of topology that deals with placing a "structure" on sets (for our purposes, the real line) in such a way that it allows one to analyze closeness, limits, infinite series, convergence, continuity, and other important concepts. The basic concepts of point-set topology like "closeness," "arbitrarily small," and so on are based on the concepts of neighborhoods and open sets.

Definition: Let $a \in \mathbb{R}$ and $\delta > 0$. A δ**-neighborhood** of a is the open interval $N_\delta(a) = (a-\delta, a+\delta)$ of radius δ centered at a. Alternate ways of writing this are

$$N_\delta(a) = \{x \in \mathbb{R} : a-\delta < x < a+\delta\} = \{x \in \mathbb{R} : |x-a| < \delta\}.$$

Important Note: Point set topology depends strongly on the ideas of set theory introduced by Georg Cantor in the late 1800s.

This brings us to the unifying concept of this section, the open set.

Definition: A subset of real numbers $A \subseteq \mathbb{R}$ is an **open set** if for every $a \in A$ there exists a $\delta > 0$ such that $N_\delta(a) \subseteq A$. That is

$$A \subseteq \mathbb{R} \text{ open} \Leftrightarrow (\forall a \in A)(\exists \delta > 0)(N_\delta(a) \subseteq A)$$

In the definition of an open set, when we say "there exists a δ greater than zero," we are normally thinking of a small δ, not a large one. Intuitively speaking, a set is open if you can "wiggle" around any point in the set and still be in the set, provided you don't wiggle too much. Another way of thinking about open sets is every point in an open set is surrounded by other points in the set and insulated from the outside.

Note: Because of the matching of real numbers with points on the number line, we often refer to real numbers as "points."

Example 1 Open Sets

a) Open intervals $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ like you studied in calculus are open sets.

b) The unbounded open intervals

$$(a,\infty) = \{x \in \mathbb{R} : x > a\}$$
$$(-\infty,b) = \{x \in \mathbb{R} : x < b\}$$

are open sets. What does it mean for a set *not* to be open? To answer that question, we negate[1] the definition of an open set. Comparing side by side the definition of an open set versus a not-open set, we have

[1] You can see the benefit of the predicate logic notion that allows one to negate sentences very easily.

$$\boxed{\begin{array}{ll} A \subseteq \mathbb{R} \text{ open} & \Leftrightarrow (\forall a \in A)(\exists \delta > 0)(N_\delta(a) \subseteq A) \\ A \subseteq \mathbb{R} \text{ not open} & \Leftrightarrow (\exists a \in A)(\forall \delta > 0)(N_\delta(a) \not\subseteq A) \end{array}}$$

In other words, not open means there exists at least one point $a \in A$ "right on the boundary" of A that is not "insulated" by neighborhoods from the outside. The interval $(a,b]$ is not open since if the point $x = b$ is wiggled *any* amount, it will be outside $(a,b]$. More formally

$$(\forall \delta > 0)\left[(b-\delta, b+\delta) \not\subseteq (a,b]\right].$$

The most important properties of open sets relate to their union and intersection.

Theorem 1: Main Theorem of Open Sets[2]

- The union of any number of open sets is open.
- The intersection of any *finite* number of open sets is open.

Proof

Union of Open Set is Open We show that if $\{A_\alpha\}_{\alpha\in\Delta}$ is an arbitrary family of open sets, then

$$\bigcup_{\alpha\in\Delta} A_\alpha$$

is an open set. To show this let

$$a \in \bigcup_{\alpha\in\Delta} A_\alpha$$

Hence a belongs to some neighborhood $N_\delta(a)$ in some member A_β of the family. Hence, we have

$$(\exists \delta > 0)\left[a \in N_\delta(a) \subseteq A_\beta \subseteq \bigcup_{\alpha\in\Delta} A_\alpha\right]$$

which implies that the union of open sets is open. ▌

Finite Intersection of Open Sets is Open

The intersection of any finite number of open sets is open.

Proof: Let $\{A_k\}_{k=1}^{n}$, be a finite family of open sets. To show that the intersection

[2]All sets in this section are subsets of the real numbers unless otherwise specified.

$$\bigcap_{k=1}^{n} A_k$$

is open, let

$$a \in \bigcap_{k=1}^{n} A_k$$

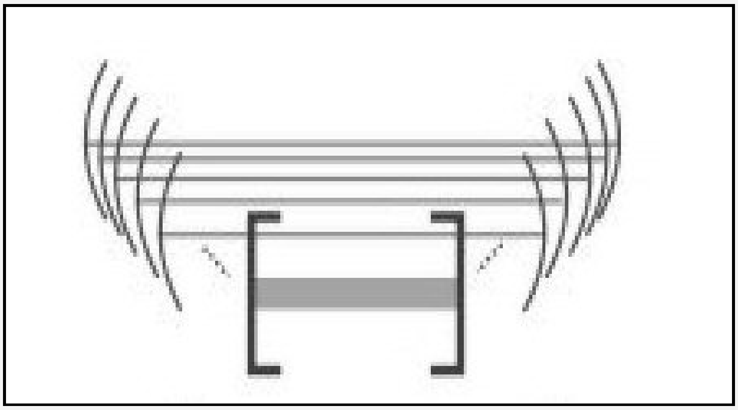
Then, since each A_k is open, we have

$$(\forall a \in A_k)(\exists \delta_k > 0)\left(a \in N_{\delta_k}(a) \subseteq A_k\right)$$

If we pick[3] $\delta = \min\{\delta_k : k = 1, 2, ..., n\} > 0$, we have

$$a \in N_\delta(a) \subseteq \bigcap_{k=1}^{n} A_k$$

which proves the result. ∎



Infinite intersections of open sets may not be open.

Example 2 Unions and Intersections of Open Sets

a) $\bigcup_{n=1}^{\infty}\left(0, 2-\frac{1}{n}\right) = (0,1) \cup \left(0, \frac{3}{2}\right) \cup \left(0, \frac{5}{3}\right) \cup \cdots = (0,2)$ (open)

b) $\bigcap_{n=1}^{N}\left(0, 1+\frac{1}{n}\right) = (0,2) \cap \left(0, \frac{3}{2}\right) \cap \cdots \left(0, 1+\frac{1}{N}\right) = \left(0, 1+\frac{1}{N}\right)$ (open)

[3] It is necessary that the number of open sets be *finite*, else the values of δ_k might *not* have a minimum value.

c) $\bigcap_{n=1}^{\infty}\left(0,1+\frac{1}{n}\right)=(0,2)\cap\left(0,\frac{3}{2}\right)\cap\left(0,\frac{4}{3}\right)\cap\cdots=(0,1]$ (not open)

Theorem 2: Characterization of Open Sets A set $A\subseteq\mathbb{R}$ of real numbers is open if and only if it is an open interval or a finite or countable union of disjoint open intervals.

Closed Sets The concept of open sets leads us to what might be called the opposite of an open set, a *closed* set.

Definition: A set $A\subseteq\mathbb{R}$ is **closed** if and only if its complement $\overline{A}$ is open.

Example 3 Closed Sets

a) The **closed intervals** $A=[a,b]$ studied in calculus are closed sets since their complements $\overline{A}=(-\infty,a)\cup(b,\infty)$ are open sets. For example, the closed intervals $[0,1],[-2,3]$ are closed sets.

b) The **unbounded closed intervals** $A=[a,\infty),B=(-\infty,b]$ are closed sets since their complements $\overline{A}=(-\infty,a)$, $\overline{B}=(b,\infty)$ are open. For example, $[0,\infty),(-\infty,0]$ are closed sets.

c) Any **singleton set** $\{a\}$ is a closed set since its complement $(-\infty,a)\cup(a,\infty)$ is open. In fact, any **finite set** $\{a_1,a_2,...,a_n\}$ is closed since its complement is the union of open intervals , which by Theorem 1 is an open set.

d) The natural numbers $\mathbb{N}$ or integers $\mathbb{Z}$ are closed sets.

Keep in mind not all sets are open or closed; the sets $A=(0,1]$ and $B=[-3,2)$ are neither open nor closed.

The set of real numbers $\mathbb{R}$ is open by the definition of an open set, and hence its complement, the empty set, is closed. But the empty set is also open vacuously by definition since there is no point $a\in\varnothing$ to check for the condition $a\in N_\delta(a)\subseteq\varnothing$. But if the empty set is open, that means $\mathbb{R}$ is closed. This means $\mathbb{R}$ and $\varnothing$ are *both* open and closed sets. In fact, they are the *only* sets of real numbers that are both open and closed. All other sets are either open, closed, or neither.

We have seen that the intersection of an arbitrary collection of open sets is open, and that the finite intersection of open sets is open. We now ask if there are corresponding properties for closed sets. The following theorem answers this question and shows the "dual" nature of this property.

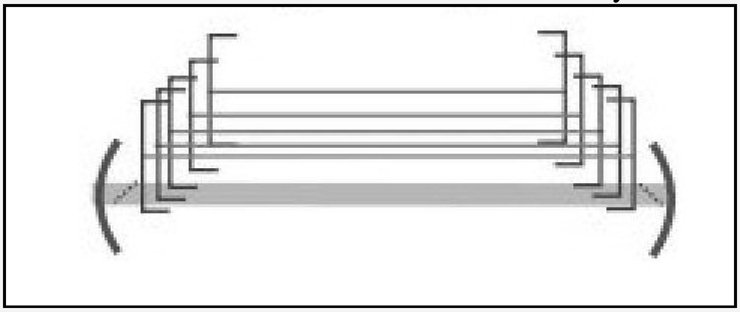
Theorem 3: Main Theorem of Closed Sets

● The intersection of an arbitrary collection of closed sets is closed.

● The union of a *finite* number of closed sets is closed.

The proof is based on DeMorgan's laws whose proof is left for the reader. See Problem 13.

Important Note: The infinite union of closed sets may not be closed.



Example 4: Unions and Intersections of Closed Sets

The following examples illustrate that the intersection of closed sets is closed, but that the union of closed sets may not be closed, unless it is the union of a finite number of closed sets.

a) $\bigcap_{n=1}^{\infty}\left[0,1+\frac{1}{n}\right]=[0,2]\cap\left[0,\frac{3}{2}\right]\cap\left[0,\frac{4}{3}\right]\cap\cdots=[0,1]$ (closed)

b) $\bigcup_{n=1}^{N}\left[0,2-\frac{1}{n}\right]=[0,1]\cup\left[0,\frac{3}{2}\right]\cup\left[0,\frac{5}{3}\right]\cup\cdots=\left[0,1-\frac{1}{N}\right]$ (closed)

b) $\bigcup_{n=1}^{\infty}\left[0,2-\frac{1}{n}\right]=[0,1]\cup\left[0,\frac{3}{2}\right]\cup\left[0,\frac{5}{3}\right]\cup\cdots=[0,2)$ (not closed)

Interior, Exterior, and Boundary of a Set

Three important concepts of topology are the concept of interior, exterior, and boundary of a set[4].

[4] We are studying basic topology of the real numbers, which allows us to talk about closeness, convergence, and so on. In general, a **topology** on a set is a family of subsets, called **open sets**, which are closed under unions and finite intersections.

Interior Point of a Set: We define a point $a \in \mathbb{R}$ to be an **interior point** of a set A if and only if there exists a $\delta > 0$ such that $a \in N_\delta(a) \subseteq A$. We denote the interior points of a set A by $\operatorname{Int}(A)$, called the **interior** of the set. In the language of predicate logic:

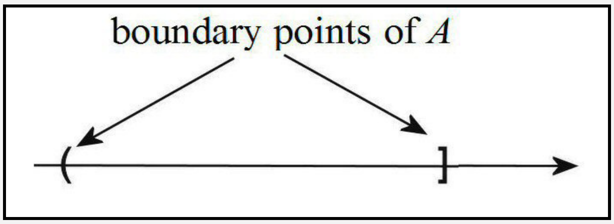
$$a \in \operatorname{Int}(A) \Leftrightarrow (\exists \delta > 0)\big(N_\delta(a) \subseteq A\big)$$

An interior point of A always belongs to A. The interior $\operatorname{Int}(A)$ of A is always open.

Boundary Point of a Set: A point $a \in \mathbb{R}$ is a **boundary point** of A if and only if for any $\delta > 0$ the δ-neighborhood of a intersects both A and the complement of A. The boundary points of A are denoted by $\operatorname{Bdy}(A)$. In the language of predicate logic, we have

$$a \in \operatorname{Bdy}(A) \Leftrightarrow (\forall \delta > 0)[(N_\delta(a) \cap A \neq \varnothing) \wedge (N_\delta(a) \cap \bar{A} \neq \varnothing)] \ .$$

A boundary point of A may or may not belong to A. The set of boundary points $\operatorname{Bdy}(A)$ of a set is always closed.



Exterior Point: A point $a \in \mathbb{R}$ is an **exterior point** of a set A if and only if there exists a $\delta > 0$ such that $a \in N_\delta(a) \subseteq \bar{A}$. We denote the set of exterior points, called the **exterior** of A, by $\operatorname{Ext}(A)$. In the language of predicate logic, we have

$$a \in \operatorname{Ext}(A) \Leftrightarrow (\exists \delta > 0)\big(a \in N_\delta(a) \subseteq \bar{A}\big)$$

An exterior point of A will never belong to A. The set of exterior points $\operatorname{Ext}(A)$ of a set is always open.

Important Note: Intuitively, a point in a set is an interior point if it is not “right on the edge” of the set. Boundary points are ponts “right on the edge” of the set. They are close to points inside and outside the set. On the other hand, the exterior points of a set are points that the set can't get "close" to.

Interiors, Boundaries, and Exteriors of Common Sets

Table 1 shows the interiors, boundaries, and exteriors of some common sets. Note that the interior and exterior of a set are always open, and the exterior, which is the complement of the union of the interior and boundary, is always closed.

$A \subseteq \mathbb{R}$	$\operatorname{Int}(A) \subseteq A$	$\operatorname{Bdy}(A)$	$\operatorname{Ext}(A)$
$\varnothing$	$\varnothing$	$\varnothing$	$\mathbb{R}$
(a,b)	(a,b)	$\{a,b\}$	$(-\infty,a)\cup(b,\infty)$
$[a,b]$	(a,b)	$\{a,b\}$	$(-\infty,a)\cup(b,\infty)$
$(a,b]$	(a,b)	$\{a,b\}$	$(-\infty,a)\cup(b,\infty)$
$\{a\}$	$\varnothing$	$\{a\}$	$(-\infty,a)\cup(a,\infty)$
$\{a,b,c\}$	$\varnothing$	$\{a,b,c\}$	$\mathbb{R}-\{a,b,c\}$
$\left\{1,\frac{1}{2},\frac{1}{3},..\right\}$	$\varnothing$	$\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...,\ 0\right\}$	$\mathbb{R}-\left\{1,\frac{1}{2},\frac{1}{3},..\ 0\right\}$
$(0,1)\cup\{2\}$	$(0,1)$	$\{0,1,2\}$	$(-\infty,0)\cup(1,2)\cup(2,\infty)$
$(-1,0)\cup(0,1)$	$(-1,0)\cup(0.1)$	$\{-1,0,1\}$	$(-\infty,-1)\cup(1,\infty)$
$\mathbb{N}$	$\varnothing$	$\mathbb{N}$	$\mathbb{R}-\mathbb{N}$
$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}$	$\mathbb{R}-\mathbb{Z}$
$\mathbb{Q}$	$\varnothing$	$\mathbb{R}$	$\varnothing$
$\mathbb{R}$	$\mathbb{R}$	$\varnothing$	$\varnothing$

Interiors, boundaries, and exteriors of sets
Table 1

Limit Points

The concept of a limit point is fundamental in the development of calculus and analysis. The reader will recall that the two fundamental operations of the calculus, the derivative and integral, are both limits. We can thank the French mathematician Augustin-Louis Cauchy (1789-1857) and German mathematician Karl Weierstrass (1815-1897) for giving the first rigorous definition of the limit, the so-called epsilon, delta (ε,δ) definition, which allowed mathematicians to argue with greater precision and avoid the imprecise and intuitive reasoning of the past.

Definition: A number a is a **limit point** of a set A if and only if for every $\delta > 0$ the δ-neighborhood of a contains a member of the set A. A limit point of a set may or may not belong to the set. We denote the set of limit points of a set A by $\text{Limits}(A)$.

Important Note: Intuitively, a limit point of a set is a point that can be "approached" by points in the set. In other words, a set likes to "snuggle up" to its limit points.

Closed Sets Contain Their Limit Points

A good way to determine if a set is closed is to show that its complement is open. Although this indirect procedure is convenient, there is a direct way to determine if a set is closed which involves the limit points of a set. The following theorem makes this precise.

Theorem 4: A subset $A \subseteq \mathbb{R}$ is closed if and only if it contains all its limit points.

Proof: $(\Rightarrow)$ We show that any $x \notin A$ is not a limit point of A and hence A contains its limit points. We assume A is closed which means its complement $\overline{A}$ is open. Hence for any $x \in \overline{A}$ there is an open neighborhood of x lying completely in $\overline{A}$. Hence x is not a limit point of A which means A contains its limit points.
$(\Leftarrow)$ If A contains its limit points, then every $x \in \overline{A}$ is not a limit point of A. Hence, there exists an open neighborhood of x lying completely in $\overline{A}$. By definition this means that $\overline{A}$ is an open set since A is closed. ▌

Table 2 gives the limit points of some common sets of real numbers. If the last column is a "yes" that means the set A is closed. (In other words, it contains its limit points,) If the answer is “no” that means the set is either open or neither open or closed.

$A \subseteq \mathbb{R}$	Limit Points(A)	Limit Points$(A) \subseteq A$?
$\varnothing$	$\varnothing$	**yes (closed)**
(a,b)	$[a,b]$	no
$[a,b]$	$[a,b]$	**yes (closed)**
$(a,b]$	$[a,b]$	no
$\{a\}$	$\varnothing$	**yes (closed)**
$\{a,b,c\}$	$\varnothing$	**yes (closed)**
$\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\right\}$	$\{0\}$	no
$(0,1)\cup\{2\}$	$[0,1]$	no
$[0,1]\cup\{2\}$	$[0,1]$	**yes (closed)**
$\mathbb{N}$	$\varnothing$	**yes (closed)**
$\mathbb{Z}$	$\varnothing$	**yes (closed)**
$\mathbb{Q}$	$\mathbb{R}$	no
$\mathbb{R}$	$\mathbb{R}$	**yes (closed)**

Limit points of some common sets
Table 2

Problems

1. Tell if the following sets subsets of $\mathbb{R}$ are open, closed, both, or neither.

   a) $(-1,0)\cup(0,1)$ Ans: open

   b) $[0,\infty)$ Ans: closed

   c) $(0,\infty)$ Ans: open

   d) $\mathbb{N}$

   e) $\mathbb{Z}$

   f) $\mathbb{Q}$

   g) $A=\{0,1,2,...,100\}$

h) $\{x : |x-1| < 3\}$

i) $\varnothing$

j) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\right\}$

k) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\right\} \cup \{0\}$

l) $\{x : x^2 > 0\}$

m) $\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 3 - \frac{1}{n}\right)$

n) $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$

o) $\bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$

p) $\bigcap_{k=1}^{\infty} \left[0, \frac{1}{k}\right]$

2. **Interiors, Boundaries, and Exteriors** Fill in the blanks in the following table.

	A	$\text{Int}(A)$	$\text{Bdy}(A)$	$\text{Ext}(A)$
a)	$\mathbb{Z}$			
b)	$\{\sin n : n \in \mathbb{N}\}$			
c)	$(0, \infty)$			
d)	$[0,1] \cup \{2\}$			
e)	$\{1, 5, 6\}$			
f)	$\{\sin x : 0 \le x \le \pi\}$			
g)	$(-1,0) \cup (0,1)$			
h)	$\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$			

3. **True or False**

a) A non empty set can be both open and closed.
b) A point can lie both in the interior and on the boundary.
c) Finite sets are always closed.
d) Infinite sets are always open.
e) The boundary of a set is always finite.

4. **Mystery Sets** Find two sets, which are not equal, but have the same interior, boundary, and exterior.

5. **Finding Examples** Find the following sets of real numbers.

a) Set with two boundary points in the set, one not in the set.
b) Set with four boundary points in the set, three not in the set.
c) A set with three boundary points, none in the set.
d) A set with three boundary points, all in the set

6. **Finite Sets Closed** Show that the finite set $A = \{1, 2\}$ is closed by finding its complement and showing the complement is an open set.

7. **Limit Points** If they exist, find the limit points of the following sets.

a) $\mathbb{N}$

b) $\mathbb{Q}$

c) $\mathbb{R}$

d) $(2,4) \cup (4,5)$

e) $\left\{(-1)^n : n \in \mathbb{N}\right\}$

f) $\varnothing$

g) $\mathbb{Q} \cap (0,1)$

h) $\left\{\dfrac{m}{2^n} : m, n \in \mathbb{N}\right\}$

i) $\left\{m + \dfrac{1}{n} : m, n \in \mathbb{N}\right\}$

8. **Closed Sets** Find the limit points of the following sets and determine if the sets are closed.

a) $\mathbb{Z}$ Ans: $\text{Limits}(\mathbb{Z}) = \varnothing$ Hence $\mathbb{Z}$ is closed.
b) $\mathbb{Q}$
c) $\mathbb{R}$
d) $(2,4)\cup(4,5)$
e) $\{(-1)^n : n \in \mathbb{N}\}$
f) $\varnothing$
g) $\mathbb{Q}\cap(0,1)$
h) $\left\{\dfrac{m}{2^n} : m,n \in \mathbb{N}\right\}$
i) $\left\{m + \dfrac{1}{n} : m,n \in \mathbb{N}\right\}$

9. **Intersections of Closed Intervals** The intersection of a finite number of closed intervals is one of three types of sets. What are they?

10. **Intersections of Open Intervals** The intersection of a finite number of open intervals is one of two types of sets. What are they?

11. **Examples** Give examples of the following.

a) A bounded set with no limit points.
b) An unbounded set with one limit point.
c) A set with two limit points.
d) An unbounded set whose limit points have cardinality $\aleph_0$.
e) An unbounded with one limit point.
f) An open set with no limit points.

12. **Sets and Limits** Find examples of a set A of real numbers with the following properties:

a) A set that is equal to its limit points.
b) A set that is a subset of its limit points.
c) A set that contains all its limit points.
d) A set that is not a subset of its limit points and its limit points are not a subset of the set.

13. **Intersections and Unions of Closed Sets** Show that the intersection of any family of closed sets is closed and that the union of a finite number of closed sets is closed. Hint: Use the properties of open sets and DeMorgan's laws.

14. **Cantor Set** Let $I = [0,1]$. Remove the open middle third

$$\left(\frac{1}{3},\frac{2}{3}\right)$$

and call A_1 the set that remains; that is

$$A_1 = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right].$$

Now remove the open third intervals from each of these two parts of A_1, and call the remaining part A_2. Thus

$$A_2 = \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right]$$

Continuing in this manner, remove the open middle third of each segment in A_k and call the remaining set A_{k+1}. Note that we will get

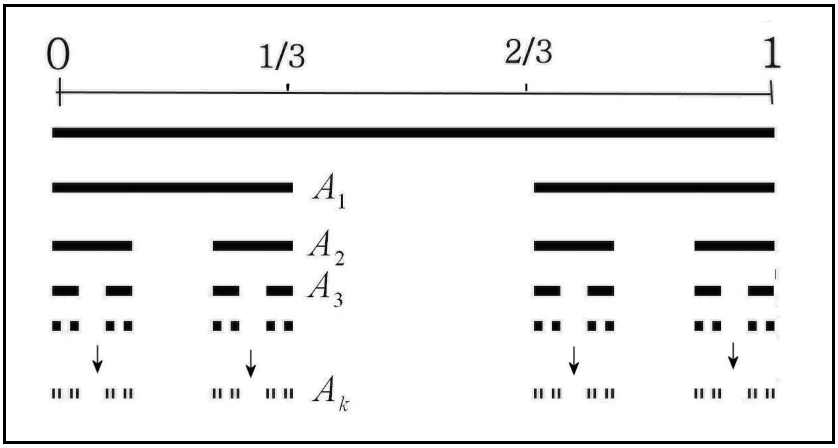
$$A_1 \supset A_2 \supset A_3 \supset \cdots A_k \supset \cdots$$

Continue this process indefinitely, always removing the middle third of existing segments. See Figure 1. The limiting set of this infinite process is called the Cantor set, and is defined as

$$C = \bigcap_{k=1}^{\infty} A_k .$$

a) Are there any points left in the Cantor set?
b) Show the Cantor set is closed[5].

[5] The Cantor set has a variety of interesting mathematical properties. It has no interior, every point in the Cantor set is a limit point. The Cantor set is uncountable, but at the same time has total "length" (measure) of zero.



Cantor set

15. **Intersection of Open Sets** Find an example of a family of open sets whose intersection is not open.

16. **Union of Closed Sets** Find an example of a family of closed sets whose union is not closed.

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