Section 6.1 Symmetries and Algebraic Systems

Purpose of Section: To introduce the idea of a symmetry of an object in the plane, which will act as an introduction to the study of the algebraic group which follows in Section 6.2.

Abstraction and Abstract Algebra

The ability to think abstractly is a unique feature of human thought, a capacity not shared by "lower forms" of living creatures[1]. The power to capture the essence of what we see and experience is so engrained in our mental processes, that we never give it a second thought. If the human mind did not have the capability to abstract commonalities in daily living, we would be living in a different world. Suppose we lacked the capacity to grasp the "essence" of what makes up a chair. We would be forced to call every chair by a different name in order to communicate to others what we are referring to. The statement "the chair in the living room" would have no meaning unless we knew exactly what chair was being mentioned. Parents point to a picture of a dog in a picture book and tell their one-year old infant, "dog", and it is a proud moment for the parent when the child sees a dog in the yard and says, "dog!"

The concept of number is a crowning achievement of human's ability to abstract the essence of size of sets. It is not necessary to talk about "three people," "three days", "three dogs" and so on. We have abstracted among those things the commonality of "*threeness*" so there is no need to say "three goats plus five goats is eight goats," or "three cats plus five cats is eight cats," we simply say three plus five equals eight.

The current chapter is a glimpse into some ideas of what is called *abstract algebra*. Before defining what we mean by an abstract algebra, you should realize you have already studied some abstract algebras whether you know it or not. The integers are an example of an abstract algebra, although you probably have never called them abstract or an algebra. The integers are a set of objects which, along with the operations of addition, subtraction, and multiplication defined on the integers, which include a collection of axioms or rules which the operations obey. Abstract algebra *abstracts* the essence of the integers and other mathematical structures, and says; "let's not study just this or that, let's study *all* things which have certain properties of interest. (Not all that different from when an infant first says "dog," realizing there are more dogs than just the one in the

---

[1] At least we humans think so.

picture book.) Abstract mathematics allows one to think about attributes and relationships, and not focus on specific objects that possess those attributes and relationships.

The benefits of abstraction are many; it uncovers relationships between different areas of mathematics by allowing one to “rise up” above the nuances of a particular area of study and see things from a broader viewpoint, like seeing the forest and not simply the trees, so to speak. The disadvantage of abstraction, if there is one, is that abstract concepts are more difficult to grasp and require more “mathematical maturity” before they can be appreciated. In summary, abstract algebra studies general mathematical structures with given properties, important structures being *groups, rings*, and *fields*.

Before we start our formal discussion of algebraic groups in the next section, we motivate their study with the introduction of symmetries.

Important Note: One hundred years ago when the ideas of abstract algebraic systems were starting to percolate into popular textbooks the subject was called “modern algebra.” However, over the years that term has become out of date and it's now it is simply called "algebra," not to be confused with the basic algebra studied in middle and high school.

Symmetries

We are all familiar with symmetrical objects, which we generally think of as objects of beauty, and although you may not be prepared to give a precise mathematical definition of symmetry, you know one when you see one. Most people would say a square is more symmetrical than a rectangle, and a hexagon more symmetrical than a square, and a circle is the most symmetrical object of all.

Regular patterns and symmetries are known to all cultures and societies. Although we generally think of symmetry in terms of geometric objects, we can also include physical objects as well, like a molecule, the crystalline structure of a mineral, a plant, an animal, the solar system, or even the universe. The concept of symmetry also embodies processes like chemical reactions, the scattering of elementary particles, a musical score, the evolution of the solar system, and even mathematical equations. In physics, symmetry[2] has to do with the invariance (i.e. unchanging) of natural laws under space and time transformations. A physical law which has space/time symmetries establishes that the law is independent of translating, rotating, or reflecting the coordinates of the system. The symmetries

[2] For the seminal work on symmetries, see *Symmetry* by Hermann Weyl (Princeton Univ Press) 1980.

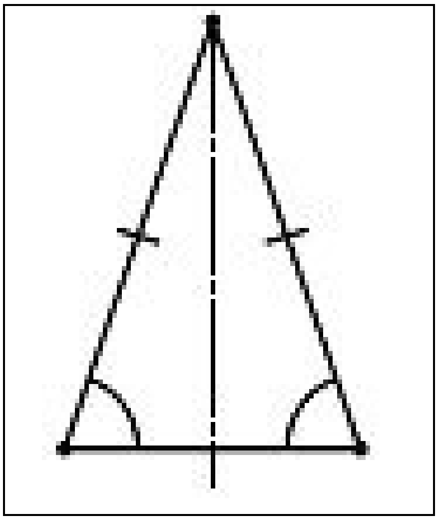
of a physical system are fundamental to how the system acts and behaves. No doubt you have seen "symmetric" equations in mathematics before. You would call the equation

$$x^2 + y^2 + z^2 + 3xyz + 1 = 0$$

symmetric in the three variables x, y, z since interchanging any two the resulting equation is unchanged.

Symmetries in Two Dimensions

For a single (bounded) figure in two dimensions, there are two types of symmetries[3]. There is symmetry across a line in which one side of the object is the mirror image of its other half. This bilateral symmetry, or the symmetry of left and right, is common in the structure of higher animals, especially in humans. This type of symmetry is called **line symmetry** (or **reflective** or **mirror** symmetry). Figure 1 shows an isosceles triangle with a line symmetry through its vertical median.



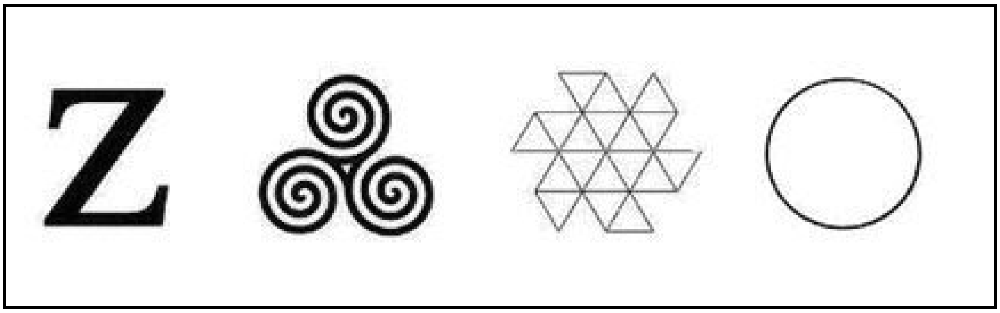
Line symmetry
Figure 1

Although the male sawfly in Figure 2 has a bilateral symmetry. other figures are more symmetric. The square has four lines (or axes) of symmetry (its horizontal and vertical midlines, and two diagonal lines), a pentagon has five lines of symmetry, and a circle has an infinite number.

[3]We do not include translation symmetries here since we are considering only bounded geometric objects.

Degrees of symmetry from the sawfly to the circle[4]
Figure 2

A second type of symmetry is **rotational** (or **radial**) **symmetry**. Three of the objects in Figure 3 have rotational symmetries about a point but no line symmetries. An object has a rotational symmetry if the object is repeated when rotated certain degrees about a central point. The letter "Z" repeats itself if rotated 0 and 180 degrees[5] so it has rotational symmetry of degree 2. The Celtic triskelion symbol is more rotationally symmetric, repeating itself when rotated 0, 120, and 240 degrees.



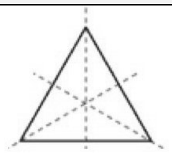
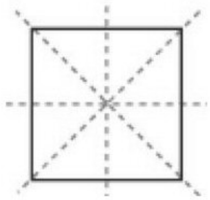
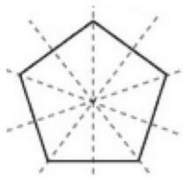
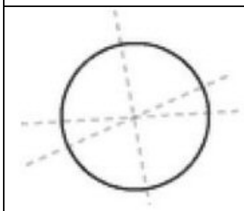
More and more rotational symmetries
Figure 3

The funny-looking figure made out of triangles repeats itself when rotated 0, 60, 120, 180, 240, and 300 degrees, so it has rotational symmetry of degree six. And finally the most symmetric of all figures in the plane, the circle which has an infinite number of both line and rotational symmetries.

Some objects have both reflective and rotational symmetries as illustrated by the regular polygons[6] in Figure 4. The equilateral triangle has 3 rotational symmetries (rotations of $0^\circ, 120^\circ$ and 240° about a center point) and 3 reflective symmetries through median lines passing through the three vertices. A regular

[4] We thank artist Joe MacGown for the drawing of the sawfly.
[5] It is a convention to call a 0 degree rotation a rotational symmetry. Hence, all objects have at least one rotational symmetry.
[6] Recall that a regular polygon is a polygon whose lengths of sides and angles are the same. Common ones are the equilateral triangle, square, pentagon, and so on.

polygon with n vertices has n rotation symmetries (each rotation $360/n$ degrees) and n lines of symmetry.

Equal Number of Rotation and Line Symmetries		
Figure	**Rotation Symmetries**	**Line Symmetries**
	3 rotations $0^\circ, 120^\circ, 240^\circ$	3 line symmetries
	4 rotations $0^\circ, 90^\circ, 180^\circ, 270^\circ$	4 line symmetries
	5 rotations $0^\circ, 72^\circ, 144^\circ, 216^\circ, 288^\circ$	5 line symmetries
	Infinite number of rotation symmetries	Infinite number of line symmetries

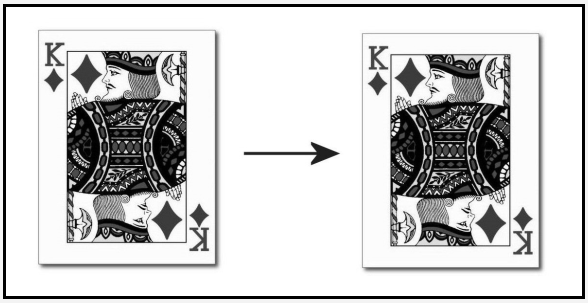
Figures having both rotational and reflective symmetries
Figure 4

Important Note: Many letters of the alphabet A,B,C,D,…have various rotation and line symmetries.

Symmetry Transformations

Although you can think of symmetries as a property of an object, there is another interpretation which is more beneficial for our purposes. A symmetry is a function or what we call in this context a mapping or transformation.

Definition A one-to-one correspondence (bijection) of an object onto itself that preserves distances is called a **symmetry** of the object. The image of an object under a symmetry transformation looks exactly like the original object. .



Rotation symmetry of 180 degrees

Important Note: There are three types of symmetries in the plane, translations, rotations, and reflections. However if an object is bounded (i.e. inside some circle with finite radius), then a translation is not a symmetry since it alters the location of the object. We now look at symmetries of some (bounded) objects in the plane.

Symmetries of a Rectangle

Figure 5 shows a rectangle where the length and width are not the same, and the corners are labeled A, B, C, D ,

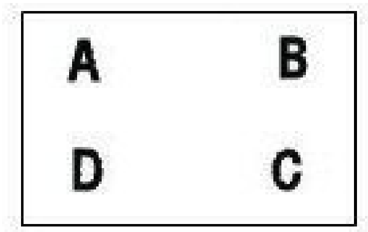
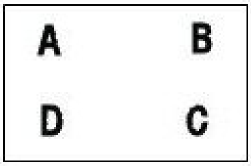
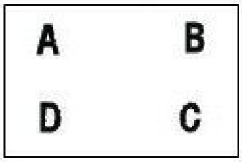
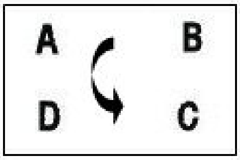
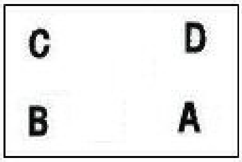
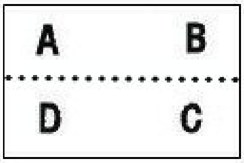
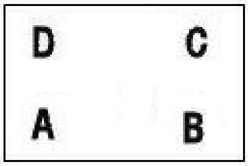
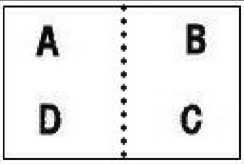
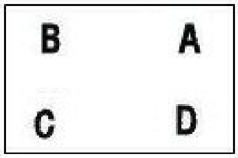


Figure 5

The rectangle has two rotational symmetries of $0^{\circ}, 180^{\circ}$ and two line (or flip) symmetries, where the lines of symmetry are the horizontal and vertical midlines. These four symmetries[7] are illustrated in Figure 6.

[7] Note that the two rotation symmetries keep the orientation the same (letters ABCD go around clockwise) while the reflection change the orientation where ABCD go around counterclockwise. Also note that to perform the reflections, one must move the 2-dimensional rectangle into 3-dimensions to perform the operation.

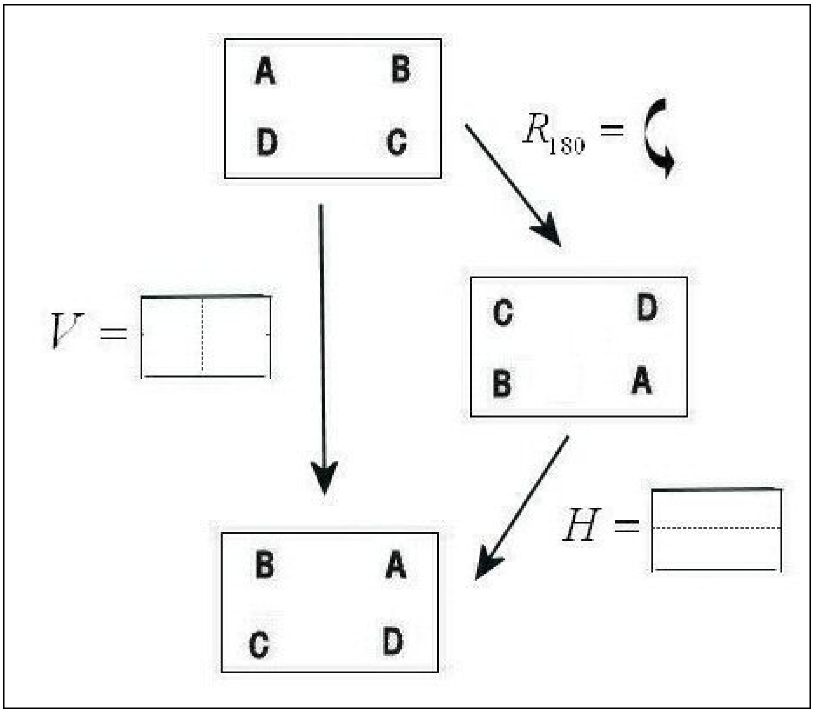
Motion	Symbol	First Position	Final Position
No Motion	e		
Rotate 180°	R_{180}		
Flip over Horizontal Median	H		
Flip over Vertical Median	V		

Four symmetries of a rectangle
Figure 6

So what do these symmetries have to do with algebraic structures? Since a symmetry is a transformation (i.e. a function) which maps the points of an object back onto itself, we can define the product of two symmetries as the composition of two symmetries. Also, since each symmetry leaves the object unchanged, so does the composition or product of two symmetries. Hence, the composition of symmetries defines a product of symmetries.

If we perform a 180° rotation[8], denoted by R_{180}, followed by H, a flip through the horizontal midline, we denote this composition or product by $R_{180}H$. The result of these two operations is illustrated in Figure 7, which we see is the same as performing the symmetry V, a flip through the vertical midline. Hence, the product $R_{180}H = V$. It is important to note that symmetry operations are carried out from "left-to-right" in the product notation.

[8] It is our convention that all rotations are done counterclockwise.



Product of symmetries $R_{180}\,H = V$
Figure 7

Note that the “do nothing” symmetry e (rotation of zero degrees), which is called the **identity symmetry**, is analogous to the number "1" in the normal multiplication of integers. Figure 8 shows the product of a few symmetries which the reader can verify.

$$eH = He = H$$
$$Ve = Ve = V$$
$$R_{180}e = eR_{180} = R_{180}$$
$$ee = e$$
$$R_{180}R_{180} = e$$
$$VV = e$$
$$HH = e$$

Products of typical symmetries
Figure 8

Also, note that two operations in a row of the symmetries $e, R_{180,}\ V,\ H$ results in returning to the original position. For that reason, we say each of these symmetries is its own inverse, which is expressed in Figure 9.

$$HH = e \Rightarrow H = H^{-1}$$
$$VV = e \Rightarrow V = V^{-1}$$
$$R_{180}R_{180} = e \Rightarrow R_{180} = R_{180}^{-1}$$
$$ee = e \Rightarrow e = e^{-1}$$

Inverses of symmetries
Figure 9

We now construct a multiplication table for all the symmetries of a rectangle, which is shown in Figure 10. The product of any two symmetries lies at the intersection of the row and column symmetries, where the row symmetry is carried out first. For example, the intersection of the row labeled R_{180} and column labeled H is V, which means $R_{180}H = V$. The borders of the cells containing the identity symmetry e are darkened as an aid in reading the table.

	e	R_{180}	H	V
e	e	R_{180}	H	V
R_{180}	R_{180}	e	V	H
H	H	V	e	R_{180}
V	V	H	R_{180}	e

Multiplication table for symmetries of a rectangle
Figure 10

Observations:

1. Every row and column of the multiplication table contains one and exactly one of the four symmetries. It is a Latin square.

2. The main diagonal contains the identity symmetry e, which means the product of each symmetry times itself is the identity, or equivalently, each symmetry is equal to its inverse.

3. The table is symmetric about the main diagonal which means the multiplication of symmetries is **commutative**. In other words, $AB = BA$, just like multiplication of numbers in arithmetic. We call this a **commutative** algebraic system.

4. The four symmetries $e, R_{180,} V, H$ of the rectangle along with the product operation form what is called an **algebraic group.** More on this in Section 6.2.

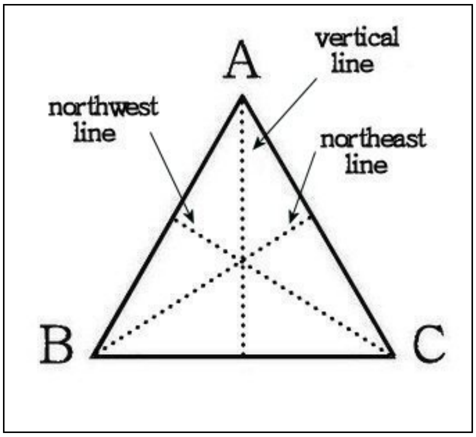
We have seen how the symmetries of a rectangle form an algebraic structure of four elements, complete with algebraic identity, and where (in this example anyway) every element has an inverse. Observe how this system is analogous to the integers with the operation of addition, with some similarities and some differences.

Important Note: OHIO is the only state in the U.S. where every letter in its name has 4 symmetries, two rotations, and two reflections.

Important Note: There are objects with no line and no rotational symmetries. Can you think of some?

Symmetries of an Equilateral Triangle

We now examine the equilateral triangle[9] which is drawn in Figure 11. It is "more symmetric" than the rectangle, having 6 symmetries, three rotational symmetries where the triangle is rotated $0^\circ, 120^\circ, 240^\circ$ about its center, and 3 line symmetries where the triangle is reflected through lines passing through the vertices as drawn as dotted line segments.



Six symmetries of an equilateral triangle
Figure 11

We denote these symmetry mappings by

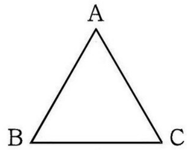
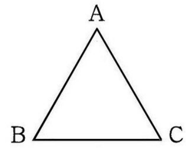
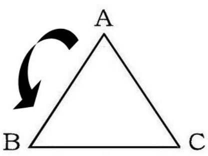
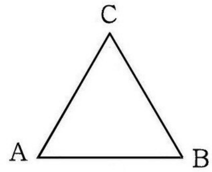
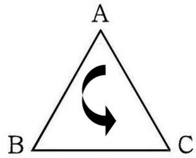
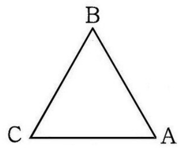
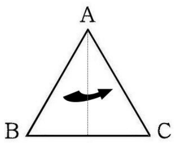
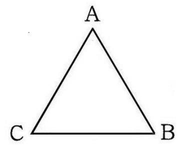
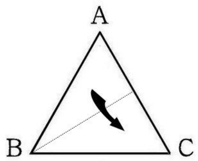
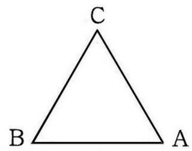
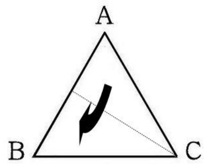
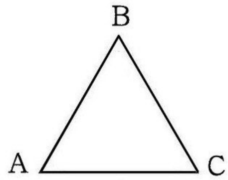
$$e, R_{120}, R_{240}, F_v, F_{nw}, F_{ne}$$

where

- $e = R_0$ is the identity (rotation by 0 degrees) symmetry
- $R_{120} =$ counterclockwise rotation of 120°
- $R_{240} =$ counterclockwise rotation of 240°
- $F_v =$ flip through the vertical median

[9] Recall that an equilateral triangle is a triangle with the three sides (or three angles) the same.

- F_{nw} = flip through the northwest median
- F_{ne} = flip through the northeast median
-

Motion	Symbol	First Position	Final Position
No Motion	$e = R_0$		
Rotate 120° Counterclockwise	R_{120}		
Rotate 240° Counterclockwise	R_{240}		
Flip through the Vertical Axis	F_v		
Flip through the Northeast Axis	F_{ne}		
Flip over the Northwest Axis	F_{nw}		

Six symmetries of an equilateral triangle
Figure 12

As we did for the rectangle, we can compute the multiplication table for the symmetries, called a Cayley table.

	$e = R_0$	R_{120}	R_{240}	F_v	F_{ne}	F_{nw}
$e = R_0$	R_0	R_{120}	R_{240}	F_v	F_{ne}	F_{nw}
R_{120}	R_{120}	R_{240}	R_0	F_{ne}	F_{nw}	F_v
R_{240}	R_{240}	R_0	R_{120}	F_{nw}	F_v	F_{ne}
F_v	F_v	F_{nw}	F_{ne}	R_0	R_{240}	R_{120}
F_{ne}	F_{ne}	F_v	F_{nw}	R_{120}	R_0	R_{240}
F_{nw}	F_{nw}	F_{ne}	F_v	R_{240}	R_{120}	R_0

Cayley table for symmetries of an equilateral triangle
Figure 13

Again, we have drawn darker around the identity symmetry $e = R_0$ as an aid in understanding and interpreting its results. We have also shaded the "northeast" and "northwest" blocks to help reading of the table.

Important Note: Some crystals in chemistry exhibit rotational symmetry but no mirror symmetry. Such shapes are called **enantiomorphic.**

Important Note: Some objects have no symmetry (other than the identity map), such as a left-hand glove or the letters G,Q, J. Take a look around you. Most objects have no symmetry at all.

Example 1: Commutative Operations Are the symmetry operations for the equilateral triangle commutative? Does the order the symmetries are performed make a difference in the outcome[10]?

Solution

We can determine if the symmetries are commutative by looking at the products in the multiplication table in Figure 11. If the table is symmetric around its diagonal elements, the symmetries are commutative. In this case, the table is not symmetric for all symmetries so the symmetry operations are not commutative. Note that $F_{nw}\, F_{ne} \neq F_{ne}\, F_{nw}$ although $R_{120}\, R_{240} = R_{240}\, R_{120} = R_0$.

Example 2: Inverse Symmetries What is the inverse of each symmetry of the equilateral triangle?

Solution Note that

[10] Not all mathematical operations are commutative. Examples are matrix multiplication and the cross product of vectors. In daily life not all operations are commutative either. Going outside and emptying the garbage is one.

$$R_0^2 = F_v^2 = F_{ne}^2 = F_{nw}^2 = R_0 \quad (R_0 = e)$$

which means the R_0, F_v, F_{ne}, F_{nv} are their own inverses. We denote this by

- $R_0^{-1} = R_0$
- $F_v^{-1} = F_v$
- $F_{ne}^{-1} = F_{ne}$
- $F_{nv}^{-1} = F_{nv}$
- $R_{120}^{-1} = R_{240}$
- $R_{240}^{-1} = R_{120}$

The fact there is exactly one identity symmetry $e = R_0$ in every row and column means that each symmetry has exactly one inverse symmetry.

Pure Mathematics: The story is told how Abraham Lincoln, failing to convince his cabinet how their reasoning was faulty, asked them, "How many legs does a cow have?" When they said four, he then continued, "Well then, if a cow's tail was a leg, how many legs does it have?" When they said five, obviously, Lincoln said, "That's where you are wrong. Just calling a tail a leg doesn't make it a let." This story may be true in the real world, but in the world of *pure mathematics* it is wrong. If you call a cow's tail a leg, then it *is* a leg. In pure mathematics, we care not what things are, only the rules or axioms that govern them.

Rotation Symmetries of Polyhedra

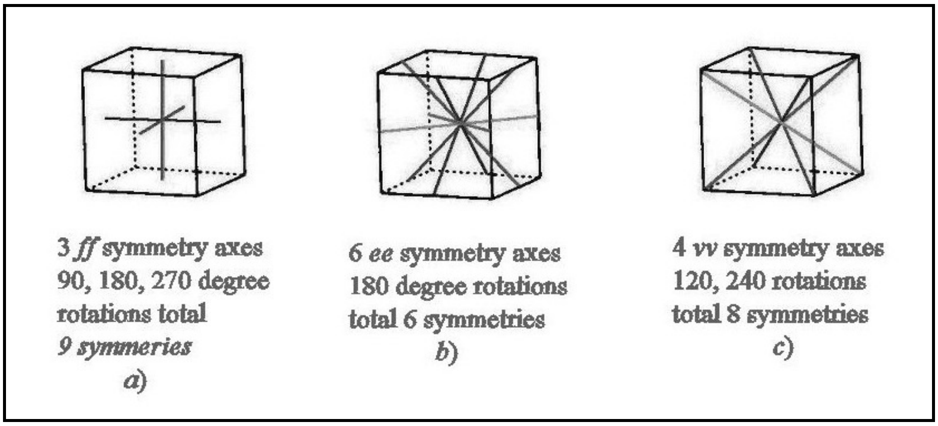
In addition to symmetries in the plane there are symmetries in higher dimensions which play an important role in many areas of science, including crystallography, which shows the number of ways atoms and molecules can be arranged within crystals. A chemist is interested in the **symmetry axes** of a polyhedron. Since polyhedra have vertices (v), edges (e), and faces (f), the symmetry axes can be one of six types vv, ee, ff, ve, vf, ef. A vv symmetry means the axis of symmetry passes through two vertices (vv), whereas a vf symmetry means the axis of symmetry passes through a vertex and an opposite face, and so on.

Rotation Symmetries of a Cube

The cube has 24 rotational symmetries of the form ff, ee, vv, meaning the axis or rotation always passes through opposite faces. edges. and vertices.

Rotation Symmetries of a Cube	**Symmetry Angles**	**Total Symmetries**
3 *ff* symmetry axes	$90^0, 180^0, 270^0$	9
4 *vv* symmetry axes	$120^0, 240^0$	8
6 *ee* symmetry axes	180^0	6
Identity map	0^0	1

These 24 symmetries are visualized in Figure 14. The reader can try to visualize these symmetries or obtain a child's block to simulate them.



Rotational symmetries of a cube
Figure 14

Problems

1. **Finding Symmetries** Find the line and rotational symmetries of the following letters of the alphabet.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

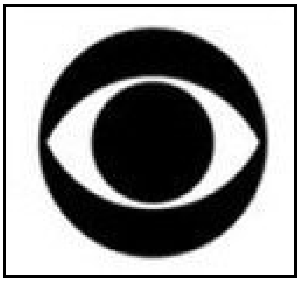
2. **Drawing Symmetries** Draw a figure that has the following symmetries.

a) 0 rotational and 0 line symmetries
b) 1 rotational and 0 line symmetries
c) 1 rotational and 1 line symmetry
d) 3 rotational and 0 line symmetries
e) 1 rotational and 0 line symmetries
f) 4 rotational and 0 line symmetries

3. **Common Symmetries** The following logos have an equal number of line and rotation symmetries. These are the symmetries of a regular n gon. Symmetries of this type are called **dihedral symmetries**. Find the rotational and line symmetries of the following logos. Don't forget the identity mapping which is a rotation of zero degrees.

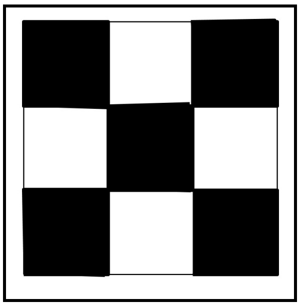
a)

b)

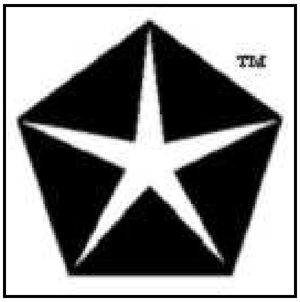


c)

d)



e)



4. **Symmetries of Solutions of Differential Equations** The solutions of the differential equation $dy/dx = y$ are the functions $y = ce^x$ where c is an arbitrary constant. Show that the transformation $x' = x + h,\ y' = y$ where h is an arbitrary real number maps the set of solutions back into the set of solutions, and hence is a symmetry transformation of the solutions of the differential equation.

5. **Symmetries of a Parallelogram** Describe the symmetries of a parallelogram.

6. **Symmetries of an Ellipse** Describe the symmetries of an ellipse

7. **Representation of D_2 with Matrices** Show that the matrices

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

with the operation of matrix multiplication obey the following multiplication table of what is called the dihedral group D_2. This means the dihedral group can be represented by matrices where the group operation is matrix multiplication.

$\circ$	e	A	B	C
e	e	A	B	C
A	A	e	C	B
B	B	C	e	A
C	C	B	A	e

Dihedral multiplication table

8. **Representation of D_3 with Matrices** Show that the following matrices with matrix multiplication satisfy the following multiplication table of the dihedral group D_3 of symmetries of a square. This means the group can be represented by matrices where the group operation is matrix multiplication.

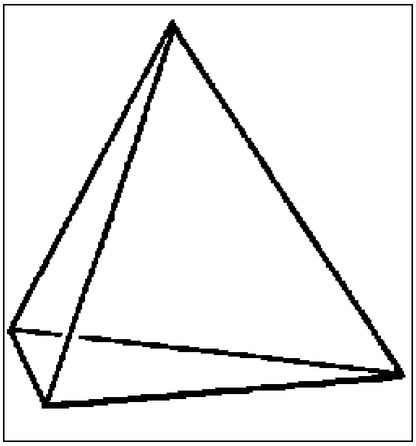
$$e=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ R_{90}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, R_{180}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},\ R_{270}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$$V=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, H=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ F_{ne}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ F_{nw}=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix},$$

9. **Symmetry Groups** Find the symmetries of the following figures and make a multiplication table for the symmetries.

a) S
b) T
c) J

10. **Symmetries of a Tetrahedron** Find the axes of symmetry and symmetries of a tetrahedron illustrated in the following figure.



11. **Cayley Table for D_3** Note that the Cayley table for the symmetries of an equilateral triangle is bunched together into four distinct blocks, two blocks consisting of rotations and two consisting of blocks of flips. From this table, tell if the following statements are true or false.

a) rotation followed by a rotation is the rotation, i.e. $RR = R$
b) rotation followed by a flip is a rotation, i.e. $RF = R$
c) rotation followed by a rotation is a flip, i.e. $RR = F$
d) rotation followed by a flip is a flip, i.e. $RF = F$
e) flip followed by a flip is a rotation, i.e. $FF = R$

f) flip followed by a rotation is a rotation, i.e. $FR = R$
g) flip followed by a flip is a flip, i.e. $FF = F$
h) flip followed by a rotation is a flip, i.e. $FR = F$
i)

ΓΣΘΨΞΠΩ