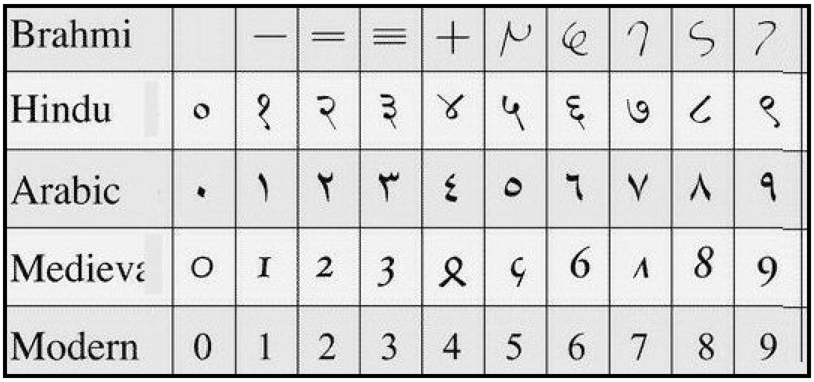
Section 4.1 Construction of the Real Numbers

Purpose of Section Starting with the nonnegative integers, we construct in order: the integers, rational numbers and the real numbers, using equivalence relations.

Introduction

At last we have reached what some might say is the Holy Grail of mathematics: the understanding of the “continuum”. While that claim may seem a little "over the top," people have, nevertheless, tried for 2500 years to understand how space and time change "continuously."

Although for practicing scientists and engineers, a continuum range of values has long been the accepted model for measurements of length, mass, and time, quantum physicists tell us that in the world of quantum physics, the continuum is replaced by the discrete. In mathematics, we do not think of the real numbers as discrete but as a continuum of values. We imagine a real variable x ranging over the real numbers assigning values to a function. A quantum physicist might argue that if mathematics were to peer closer and closer into the continuum of the real numbers, strange things may be observed, just as physicists discovered in the physical world. It is the purpose of this section and Section 4.2 to ask just that question. What *are* the real numbers, and what happens when we look at them ... up close.



The evolution towards real numbers is as old as human culture[1].

The journey from the natural numbers 1,2,3,…to the real numbers is an adventure that took humankind thousands, if not tens of thousands, of years.

[1] We thank www.archemedes.org for the use of this table.

Although no one knows for certainty when the concept of 'number" arose, a good guess might be when people started keeping track of one's possessions.. Fractions came later as refinements in counting, like 1/2 a container, or 1/3 the length of a string, which were known to Greek geometers. Although Greek mathematicians routinely used rational[2] numbers, they did not accept zero[3], or negative numbers as legitimate numbers. To them, numbers represented something and 0 and negative numbers did not represent something. Even much later prominent Renaissance mathematicians as Cardano in Italy, and Viete in France called negative numbers “absurd” and “fictitious.” But gradually, mathematicians realized the need to enlarge their thinking and address paradoxes like the meaning of $2-5$, or the solution of $x+3=1$.

The last step in the evolution of real numbers took considerably more thought. One of the great mathematical achievements of the 19th century, was our understanding of what we call the “real numbers”: those numbers which can be expressed in decimal notation, whether the decimal digits stop, go on forever in a repeating pattern, or go on in a pattern that never repeats. So now, we arrive at this foundation of real analysis, the real numbers. So what are they?

There are two basic ways we can define the real numbers. First, we can play God and bring the laws down from the mountaintop, where we say, *Here they are, the real numbers.”* This approach would be called the *synthetic* approach, whereby we list a series of axioms, which we feel embody what we think a “continuum” should be.

On the other hand, we can “construct” the real numbers, much like a carpenter builds a house. In this approach, we begin with the foundation of the natural numbers $1,2,3,\ldots$.then doing some “mathematical carpentry” construct the real numbers step by step, from the natural numbers -- to the integers -- to the rational numbers -- all the way to the real numbers. It is this “construction” approach that we carry out in this section. The synthetic axiomatic approach of "here they are" will be presented out in Section 4.2.

The Building of the Real Numbers

The construction of the real numbers begins with the simplest numbers, the natural numbers

$$\mathbb{N}=\{1,2,3,\ldots\}$$

then, by a series of steps we construct the integers

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[2] The word rational is the adjective form of ratio.

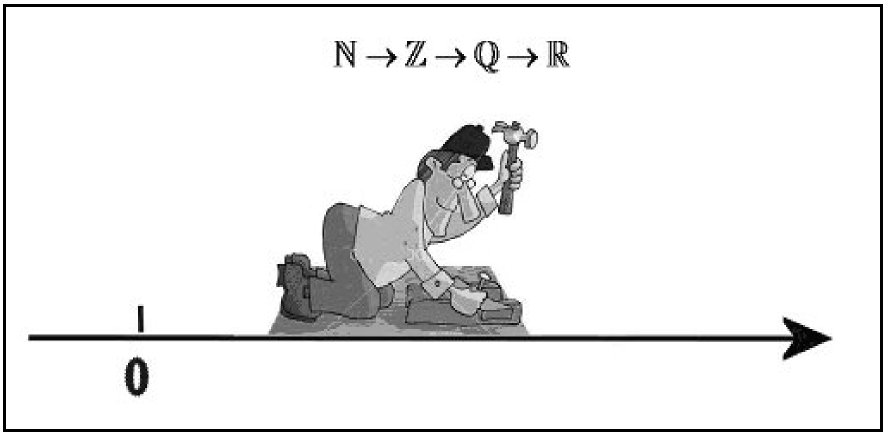
[3] [3] The first occurrence of a symbol used to represent 0 goes back to Hindu writings in India in the 9th century.

$$\mathbb{Z} = \{....-3,-2,-1,0,1,2,3,...\}$$

followed by the rational numbers

$$\mathbb{Q} = \{ p / q : p,q \in \mathbb{Z}, q \neq 0\}$$

and finally, the real numbers $\mathbb{R}$.



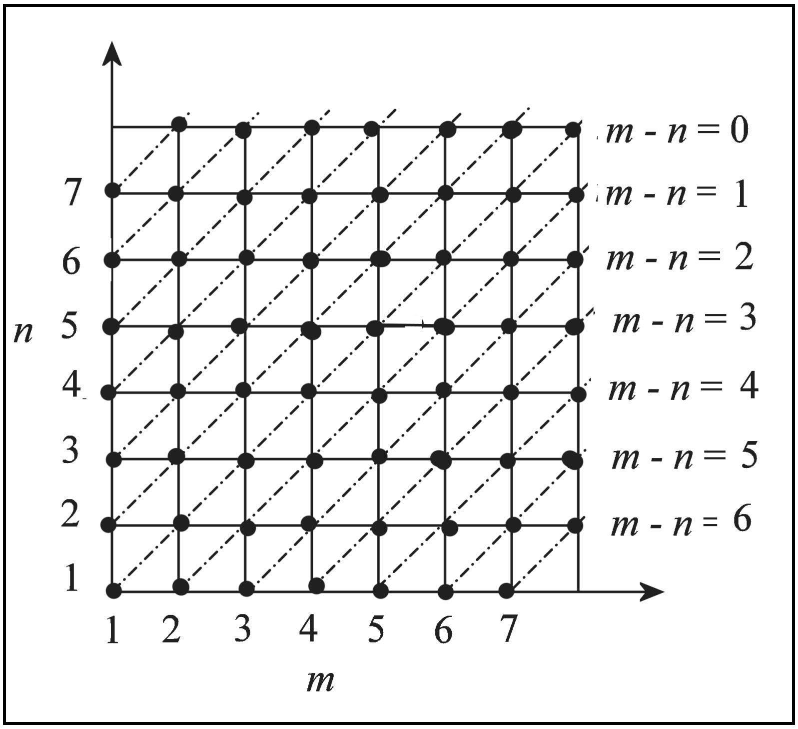
Important Note: The English mathematician/philosopher Bertrand Russell once said it must have taken many ages for humans to realize that a pair of pheasants and a couple of days were both instances of the number two. But what of the pheasants themselves? Do they understand that as a pair they represent the number two?

Construction of the Integers: $\mathbb{N} \to \mathbb{Z}$

So how does one "construct" the integers $0,\pm1,\pm2,...$ from the natural numbers $1,2,3,...$? The idea is define integers as *pairs* of nonnegative integers, where we "think" of each pair (m,n) as the difference $m-n$, thus the pair $(2,5)$ of natural numbers is defined as -3. If we then define addition, subtraction, and multiplication of the pairs (m,n) consistent with the arithmetic of the natural numbers, we have successfully defined the integers in terms of the natural numbers. To carry out this program, we begin with the Cartesian product

$$\mathbb{N}\times\mathbb{N} = \{(m,n) : m,n = 1,2,...\}$$

of pairs of natural numbers, called *grid* points, which are illustrated in Figure 1



Partitioning $\mathbb{N}\times\mathbb{N}$ into equivalence classes
Figure 1

We now define an equivalence relation " ~ " between pairs of natural numbers by

$$(m,n)\sim(m',n') \Leftrightarrow m+n'=m'+n$$

We would like to define the equivalence relation in the more suggestive form

$$(m,n)\sim(m',n') \Leftrightarrow m-n=m'-n'$$

but we can't since we don't (yet) know what a minus sign means.

We leave it to the reader to prove that this relation is an equivalence relation (See Problem 1). We learned in Section 3.3 that equivalence relations partition a set into disjoint equivalence classes. The given equivalence relation partitions the grid points $\mathbb{N}\times\mathbb{N}$ as shown in Figure 1 into 45 degree lines of the form $n=m+k,\ k=0,\pm1,\pm2,...$ A few equivalences classes are listed in Table 1, along with a designated representative $\overline{(\ )}$ from each equivalence class. Each one of these equivalence classes will define an *integer*, zero, positive or negative. For example, the equivalence class $\overline{(0,0)}$ defines zero, $\overline{(1,2)}$ defines -1 and so on.

Equivalence Class	Integer Correspondence
$\overline{(1,3)}=\{(1,3),(2,4),(3,5)\ldots\}$	-2
$\overline{(1,2)}=\{(1,2),(2,3),(3,4)\ldots\}$	-1
$\overline{(1,1)}=\{(1,1),(2,2),(3,3)\ldots\}$	0
$\overline{(2,1)}=\{(2,1),(3,2),(4,3)\ldots\}$	1
$\overline{(3,1)}=\{(3,1),(4,2),(5,3)\ldots\}$	2

Five Equivalence Classes
Table 1

We now *define* the integers $\mathbb{Z}$ as the set of equivalences classes of the quotient set of $\mathbb{N}\times\mathbb{N}$ modulo $\sim$. In other words

$$\mathbb{Z}=\left\{\ldots\ \overline{(1,4)},\overline{(1,3)},\overline{(1,2)},\overline{(1,1)},\overline{(2,1)},\overline{(3,1)},\overline{(4,1)},\ldots\right\}$$

which we call

$$\mathbb{Z}=\{\ \ldots -3,-2,-1,0,1,2,3,\ldots\}$$

More specifically,

$$\overline{(k,1)}=k-1 \qquad \text{(positive integers)}$$
$$\overline{(1,1)}=0 \qquad \text{(zero)}$$
$$\overline{(1,k)}=1-k \qquad \text{(negative integers)}$$

We now must define addition, subtraction, and multiplication for integers, consistent with the arithmetic of the natural numbers. We define for $p,q,r,s\in\mathbb{N}$

addition $\oplus$, subtraction $\ominus$, and multiplication $\otimes$

of integers (represented by pairs of natural numbers) by:

- **Addition:** $\overline{(p,r)}\oplus\overline{(q,s)}=\overline{(p+q,r+s)}$
- **Subtraction:** $\overline{(p,q)}\ \ominus\ \overline{(r,s)}=\overline{(p+s,q+r)}$
- **Multiplication**: $\overline{(p,q)}\otimes\overline{(r,s)}=\overline{(pr+qs,\ ps+qr)}$

For example

- **Addition:** $\overline{(3,5)}\oplus\overline{(1,4)}=\overline{(4,9)}$ or -5
- **Subtraction:** $\overline{(3,6)}\ \ominus\ \overline{(2,7)}=\overline{(10,8)}$ or $+2$

- **Multiplication**: $\overline{(1,3)} \otimes \overline{(7,2)} = \overline{(13,23)}$ or -10

When $p > r$ and $q > s$ the pairs (p,r) and (q,s) correspond to $p-q$ and $r-s$, hence, the above definition of addition, subtraction, and multiplication of integers, reduce to identities of the natural numbers.

Construction of the Rationals: $\mathbb{Z} \to \mathbb{Q}$

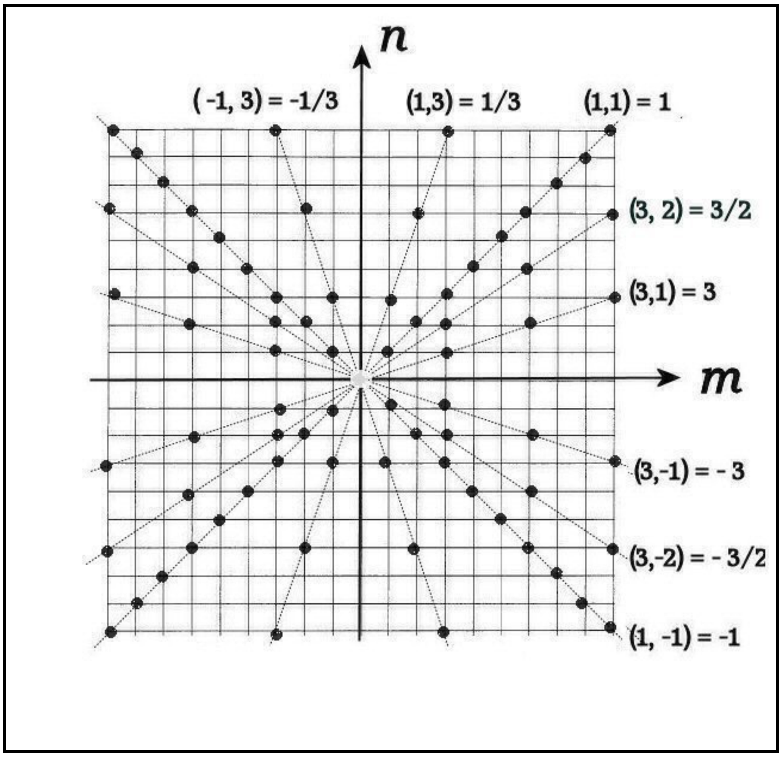
We now move on to the construction of the rational numbers $\mathbb{Q}$ from the integers, which we do by defining them in terms of pairs (m,n) of integers $\mathbb{Z}$. For example, we will define the pair of integers (m,n) as m/n. Starting with the grid points $\mathbb{Z}\times(\mathbb{Z}-\{0\})$ shown in Figure 2, we define the relation

$$(m,n) \sim (m',n') \Leftrightarrow mn' = m'n$$

which the reader can show in Problem 2, it is an equivalence relation on $\mathbb{Z}\times(\mathbb{Z}-\{0\})$. For example

$$(1,2) \sim (2,4) \sim (-3,-6) \sim (13,26) \sim (-5,-10)$$

The equivalence classes are illustrated visually as grid points on straight lines passing through the origin. See Figure 2.



Equivalence classes defining rational numbers
Figure 2

A few equivalence classes with designated representative are

Equivalence Class	**Rational Correspondence**
$\overline{(1,2)}=\{(1,2),(2,4),(3,6)\ldots\}$	$\frac{1}{2}$
$\overline{(1,-1)}=\{(1,-1),(2,-2),(3,-3)\ldots\}$	-1
$\overline{(3,-5)}=\{(3,-5),(-3,5),(6,-10)\ldots\}$	$-\frac{3}{5}$
$\overline{(0,1)}=\{(0,1)(0,2),(0,3)\ldots\}$	0

Five equivalence classes in $\mathbb{Z}\times(\mathbb{Z}-\{0\})$

Table 2

We now define the rational number p/q in terms of integers as

$$\frac{p}{q}=\overline{(p,q)}, \quad p,q\in\mathbb{Z},\ q\neq 0\,.$$

It is now necessary to define the arithmetic operations on the newly formed rational numbers. We define

- **Addition:** $\overline{(p,q)} \oplus \overline{(r,s)}=\overline{(ps+qr,qs)}$
- **Subtraction:** $\overline{(p,q)} \ominus \overline{(r,s)} =\overline{(ps-qr,qs)} \quad ps\geq rq$
- **Multiplication:** $\overline{(p,q)} \otimes \overline{(r,s)}=\overline{(pr,qs)}$

For example

- **Addition:** $\overline{(1,2)}\oplus\overline{(2,10)}=\overline{(14,20)}$ or $\frac{14}{20}=\frac{7}{10}$
- **Subtraction:** $\overline{(3,6)} \ominus \overline{(1,4)}=\overline{(6,24)}$ or $\frac{6}{24}=\frac{1}{4}$
- **Multiplication**: $\overline{(3,6)} \otimes \overline{(1,4)} =\overline{(3,24)}$ or $\frac{3}{24}=\frac{1}{8}$

The Dedekind Cut: The Rationals to the Reals

We now come to a fork in the road. There are basically two different ways to define the real numbers in terms of the rational numbers, and each has its advantages and disadvantages. On one hand, we could define the real numbers as all positive and negative decimal expansions, with repeating and non-repeating digits, the numbers with repeating decimals can be shown to be the existing rational numbers, but the decimal expansions that do not repeat define new numbers called irrational numbers. The union of the rational and irrational numbers defines the real numbers.

The disadvantage of the decimal definition is they don't relate to points on a continuum, which is what we want to investigate. There are two approaches for defining the real numbers (apart from te decimal expansion strategy), one due to Cantor and the other due to his good friend Richard Dedekind. Cantor's approach defines real numbers limits of sequences of rational numbers, like in the definition of the real number π:

$$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ 3.141592,... \to \pi$$

Although this approach has some intuitive appeal, it also demands that the reader have some background in sequences, convergence, null sequences, and other concepts from real analysis, not introduced in this book. Hence, we follow the approach of Richard Dedekind.

Historical Note: The German mathematician **Richard Dedekind** (1831-1916) was one of the greatest mathematicians of the 19th century. He made major contributions to number theory and abstract algebra. His invention of ideals in ring theory and his contributions to algebraic numbers, fields, modules, lattice, etc were crucial in the development of abstract algebra. His book '*Was sind and was sollen die Zahlen*?' (What are numbers and what should they be?) laid the foundation for the real number system and was a milestone in the history of mathematics.

Although the rational numbers have many desirable properties, they have (from the perspective of doing calculus) the undesirable property that they contain gaps. For example, there are gaps at $\sqrt{2}$ and π. In fact there are an infinite number of gaps, in fact an *uncountable* number of gaps. The idea is to "fill in" those gaps, getting a new number system which we envision as points on a continuum.

How Dedekind Cuts Define the Irrational Numbers

Dedekind's idea appeals to our intuitive grasp of the rational numbers aligned on a line. His idea was to partition the rational numbers into two disjoint sets L and U satisfying the following condition..

Definition: A **Dedekind Cut**, denoted by (L,U), is a partition of the rational numbers into two nonempty subsets L and U such that all members of the so-called **lower set** L are less than all members of **upper set** U. Stated more analytically, a Dedekind cut (L,U) is a pair of subsets L,U of $\mathbb{Q}$ satisfying

- $L \cup U = \mathbb{Q},\ L \cap U = \varnothing,\ L \neq \varnothing,\ U \neq \varnothing$
- $l \in L$ and $u \in U \Rightarrow l < u$

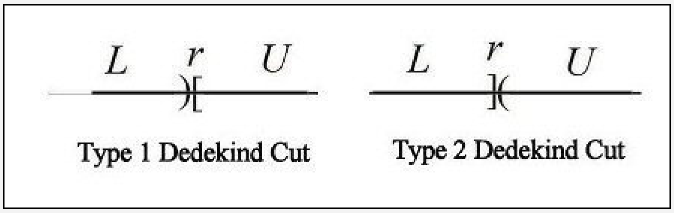
The above definition of a Dedekind cut does not uniquely define a single partition, but three different partitions or Dedekind cuts. They are

Type 1 Dedekind cut: (L,U)**:** The lower set L does not have a largest element, but the upper set U has a smallest member. That is, for a given rational number r, the cut at r is

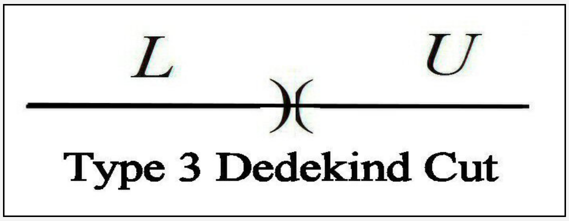
$$L = \{l \in \mathbb{Q} : l < r\},\ U = \{u \in \mathbb{Q} :\ r \leq u\}$$

Type 2 Dedekind cut (L,U)**:** The lower set L has a largest element but the upper set U does not have a smallest member. That is, for a given rational number r, the cut at r is

$$L = \{l \in \mathbb{Q} : l \leq r\},\ U = \{u \in \mathbb{Q} :\ r < u\}$$



Type 3 Dedekind cut (L,U)**:** The lower set L does not have a largest element and the upper set U does not have a smallest element.



Each Dedekind cut of Type 1 (or Type 2) corresponds to a rational number, the rational number being smallest element in the upper set in Type 1 cuts, and the largest element in the lower set in cuts of Type2. The Dedekind cuts of Type 1 and Type 2 are equivalent, both corresponding to the rational numbers.

On the other hand, the collection of Dedekind cuts of Type 3 corresponds to new numbers defined as the irrational numbers. Loosely, we say the irrational numbers fill the "gap" between the lower and upper sets of rational numbers of Type 3 Dedekind cuts. The set of real numbers, the rational and irrational numbers, is the union of the Dedekind cuts of Type 1 and Type 3 (or equivalently Type 2 and Type 3).

Example 1: Dedekind Cuts Defining Rational and Irrational Numbers

a) The Dedekind Cut (L,U) where

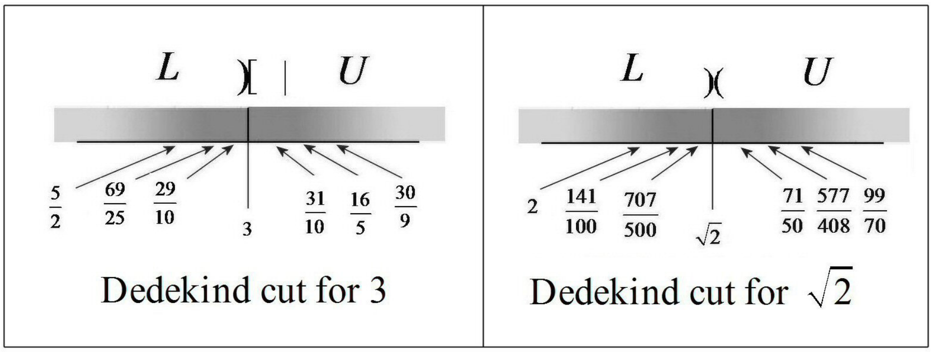
- $L=\{l\in\mathbb{Q}:l<3\}$
- $U=\{u\in\mathbb{Q}:3\le u\}$

defines the rational number 3.

b) The Dedekind Cut (L,U) where[4]

- $L=\{l\in\mathbb{Q}:l^2<2 \text{ or } l<0\}$
- $U=\{u\in\mathbb{Q}:2<u^2 \text{ and } u>0\}$

defines the irrational number $\sqrt{2}$. Figure 3 graphically illustrates these cuts.



Dedekind cuts for 3 and $\sqrt{2}$
Figure 3

[4] Although we don't prove it here, this partition of the rational numbers can be proven to satisfy the conditions of a Dedekind cut.

Ordering Dedekind Cuts

When we think of the real numbers we often imagine them as points on a number line where they are ordered according to size. The following relation defines a linear order of the real numbers.

Ordering Dedekind Cuts: For (A,B) and (C,D) Dedekind cuts, we say (A,B) is strictly less than (C,D) if A is a proper subset of C. Also we say (A,B) is less than or equal to (C,D) if A is a subset of C. We write these inequalities of Dedekind sets as

- $(A,B) < (C,D)$ if $A \subset C$
- $(A,B) \le (C,D)$ if $A \subseteq C$

This ordering is a total order on the set of real numbers, and the real numbers with this order is called **linearly ordered set**. .

We now arrive at the final destination, the definition of the real numbers as painstakingly constructed from $\mathbb{N} \to \mathbb{R} \to \mathbb{Q} \to \mathbb{R}$.

Definition of the Real Numbers (via Dedekind Cuts) The **real numbers**, denoted by $\mathbb{R}$, is the union of the rational and irrational numbers, defined by their respective Dedekind cuts along with the pervious ordering.

Arithmetic of Dedekind Cuts

Now that we have created the real numbers, complete with both rational and irrational numbers and an order relation, the next step is to carry out basic arithmetic operations like we did in grade school. We start with the two most basic numbers of arithmetic, the additive and multiplicative identities 0 and 1, which in terms of Dedekind cuts of Type 1 are the cuts having lower sets $L_0 = \{l \in Q : l < 0\}$ and $L_1 = \{l \in Q : l < 1\}$. Clearly $0 < 1$ since $L_0 < L_1$

Here are some of the basic rules of arithmetic, Dedekind cut style.

Negative of a Set: If A is any set of rational numbers, then $-A$ denotes the set of negatives of those rational numbers. If (A,B) is a Dedekind cut, then $-(A,B)$ is defined to be the Dedekind cut $(-B,-A)$.

Negative of a Dedekind Cut: A Dedekind cut[5] (A, B) is called positive if $0 \in A$ and negative if $(0 \in B)$. If (A, B) is positive, then $-(A, B)$ is negative. Likewise, if (A, B) is negative, then $-(A, B)$ is positive.

Addition of Dedekind Cuts: If (L_1, U_1) and (L_2, U_2) are Dedekind cuts, then their sum is defined by

$$(L_3, U_3) = (L_1, U_1) \oplus (L_2, U_2)$$

where

$$L_3 = \{x + y : x \in L_1, y \in L_2\}, \ U_3 = \{x + y : x \in U_1, y \in U_2\}$$

Multiplication of Dedekind Cuts: Multiplication of two factors can get tricky if one or both factors are negative. If both factors (L_1, U_1) and (L_2, U_2) are non-negative, then their product is defined by

$$(L, U) = (L_1, U_1) \otimes (L_2, U_2)$$

where

$$1 \leftrightarrow (L_1, U_1), \ L_1 = \{l \in L_1 : l < 1\}, \ \ U_1 = \{u \in U_1 : 1 \leq u\}$$
$$L = \{xy : x \in L_1, y \in L_2\}, U = \{xy : x \in U_1, y \in U_2\}$$

and at least one of the x and y is non-negative.

Problems

1. **Equivalence Relation I** Show that the relation $\sim$ defined by

$$(m, n) \sim (m', n') \text{ if and only if } m + n' = m' + n$$

between pairs of natural numbers (m, n) and (m', n') is an equivalence relation.

2. **Equivalence Relation II** Show that the relation $\sim$ defined by

$$(m, n) \sim (m', n') \Leftrightarrow mn' = m'n$$

between pairs of integers (m, n) and (m', n') is an equivalence relation.

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[5] The set A is meant to be the lower set which we formerly denoted by L and B as the upper set which we formerly denoted by U .

3. **Arithmetic in** $\mathbb{Z}$ We defined the integers in terms of equivalence classes of pairs of natural numbers (m,n). When we see $(3,5)$ we think "-2", when we see $(6,3)$ we think "+3" and so on. Perform the following arithmetic on equivalence classes of natural numbers.

a) $\overline{(1,5)} \oplus \overline{(3,2)}$ Ans: $\overline{(1,5)} \oplus \overline{(3,2)} = \overline{(4,7)} \approx -3$

b) $\overline{(1,5)} \ominus \overline{(3,2)}$ Ans: $\overline{(1,5)} \ominus \overline{(3,2)} = \overline{(3,8)} \approx -5$

c) $\overline{(1,5)} \otimes \overline{(3,2)}$ Ans: $\overline{(1,5)} \otimes \overline{(3,2)} = \overline{(13,17)} \approx -4$

4. **Arithmetic in** $\mathbb{Q}$ **and** $\mathbb{Z}$ We defined the rational numbers $\mathbb{Q}$ as equivalence classes of pairs of integers (m,n). When we see $(3,5)$ we think "3/5", when we see $(6,3)$ we think "6/3 = 2" and so on. Perform the following arithmetic steps on pairs of integers.

a) $\overline{(1,5)} \oplus \overline{(3,2)}$ Ans: $\overline{(1,5)} \oplus \overline{(3,2)} = \overline{(17,10)} \approx \dfrac{17}{10}$

b) $\overline{(1,2)} \ominus \overline{(3,2)}$

c) $\overline{(1,5)} \otimes \overline{(3,2)}$

5. **Decimal to Fractions** Find the fraction for each of the following numbers in decimal form where the bar over the numbers means the numbers are repeated.

a) 0.9999…. $\left(0.\overline{9}\right)$

b) 0.23232323…. $\left(0.\overline{23}\right)$

c) 0.0123123123… $\left(0.0\overline{123}\right)$

d) 0.001111… $\left(0.00\overline{1}\right)$

Arithmetic of Dedekind Cuts

6. **Sum of Dedekind Cuts** What are Dedekind cuts that correspond to 5 and 7 respectively and find the sum of these cuts?

7. **Product of Positive Dedekind Cuts** What are Dedekind curs that correspond to 5 and 7 and find the product of these cuts?

8. **Ordering Dedekind Cuts** What are the Dedekind cuts that correspond to 2 and π. Prove that $2 < \pi$ Choose the Type 1 cut for 2.

9. **Multiplicative Identity** Show that $1 \times 3 = 3$ in terms of Dedekind cuts.

10. **Additive Identity** Show that $7 \times 0 = 3$ in terms of Dedekind cuts.

ΓΣΘΨΞΠΩ