Section 4.2 *The* Complete Ordered Field: The Real Numbers

Purpose of Section We present the axiomatic definition of the real numbers as a **complete ordered field**. We carry out this three-step process by first defining an **algebraic field**, then introduce an order on the algebraic field, yielding an **ordered field**, and then the **completeness axiom** which allows us to define the complete ordered field, which defines the real numbers.

Introduction

Advances in function theory in the 19th century demanded a deeper understanding of the real numbers, which led to a "rigorization" of analysis by such mathematical greats as Cauchy, Abel, Dedekind, Dirichlet, Weierstrass, Bolzano, Frege, Cantor, and others. A deeper understanding of functions required precise proofs which in turn required the real number system be placed on solid mathematical ground.

Although we generally think of real numbers as points on a continuous line that extends endlessly in both directions, the goal of this chapter is to strip away everything you know about the real numbers and start afresh. This is not easy since all knowledge and mental imagery of the real numbers created over a lifetime is firmly entrenched in our minds. But if the reader is willing to wipe the slate clean and start anew, we will introduce you to a *new* mathematical entity, known by mathematicians as the "*complete, ordered field",* which, for lack of another name, we call $\mathbb{R}$. By building the axioms of the real numbers, you will have a deeper understanding of them than simply as "points on a very long line."

There are three types of axioms required to define the real numbers. First, there are the *arithmetic* axioms, called the **field axioms**, which provide the rules for adding, subtracting, multiplying and dividing. Second, there are the **order axioms***,* which allow us to compare sizes of real numbers like $2<3$, $4>0$ and $-3<0$, and so on. And last, there is an axiom, called the **continuity axiom**, which gives the real numbers that special quality which allows us to think of real numbers as "flowing" continuously with no gaps.

So let us begin our quest to define the holy grail of real analysis.

….and let there be real numbers

Arithmetic Axioms for Real Numbers

We begin by defining a set $\mathbb{R}$, but don't think of $\mathbb{R}$ as the real numbers *yet.* We begin by defining two functions from $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, one called the *addition* function and the *multiplication* function. The addition function assigns to a pair (a,b) of numbers in $\mathbb{R}$ a new element of $\mathbb{R}$ called the sum of a and b and denoted by $a+b$. The multiplication function assigns to each pair of elements in $\mathbb{R}$ a new element in $\mathbb{R}$ called the *product* of a and b and denoted by $a\times b$ or more often simply ab. These operations are called **closed operations** since when $a,b\in\mathbb{R}$ so are $a+b$ and ab.

These axioms have passed the test of time and are now chiseled in stone in the laws of mathematics and form an algebraic system called a **field**[1] (or an **algebraic field**), which is summarized as follows.

[1] Modern algebra or abstract algebra, which is distinct from elementary algebra as taught in schools, is a branch of mathematics that studies algebraic structures, such as groups, rings, fields, modules, vector spaces and other algebraic structures.

Field Axioms

A **field** is a set, which we call $\mathbb{R}$, with two binary operations, called $+$ and $\times$, where for all *a, b* and *c* in $\mathbb{R}$, the following axioms hold[2].

Addition Axioms	Name of Axiom
$(\forall a,b\in \mathbb{R})[a+(b+c)=(a+b)+c]$	associativity of addition
$(\forall a,b\in \mathbb{R})(a+b=b+a)$	addition commutes
$(\exists!0\in \mathbb{R})(\forall a\in \mathbb{R})(a+0=0+a=a)$	unique additive identity
$(\forall a\in \mathbb{R})(\exists!(-a)\in \mathbb{R})[a+(-a)=(-a)+a=0]$	unique additive inverse
Multiplication Axioms[3]	**Name of Axiom**
$(\forall a,b,c\in \mathbb{R})[a(bc)=(ab)c]$	associativity of multiplication
$(\forall a,b\in \mathbb{R})(ab=ba)$	multiplication commutes
$(\exists!1\in \mathbb{R})(\forall a\in \mathbb{R})(a\cdot 1=1\cdot a=a)$	unique multiplicative identity
$(\forall a\in \mathbb{R})[a\neq 0\Rightarrow(\exists!a^{-1}\in \mathbb{R})(a\cdot a^{-1}=a^{-1}\cdot a=1)]$	unique multiplicative inverse
Distributive Axiom	**Name of Axiom**
$(\forall a,b,c\in \mathbb{R})[a(b+c)=ab+ac]$	multiplication distributes over addition

Conventions and Notation:

In addition to the above axioms, we make the following conventions;

1. The associative axioms for both addition and multiplication tell us it doesn't matter where the parentheses are placed. In other words, we can write $a+b+c$ for $a+(b+c)$ or $(a+b)+c$. The same associative law holds for multiplication, which allows us to write $abc=a(bc)=(ab)c$.

2. The unique additive inverse of an element a is denoted by $-a$. Hence, we have $a+(-a)=0$. The multiplicative inverse of a is denoted by $a^{-1}$and often written $1/a$. Hence, $aa^{-1}=a(1/a)=1$.

Two other operations of *subtraction* and *division* can be defined directly from addition and multiplication by

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[2] We call the field $\mathbb{R}$ since we are concentrating on the real numbers, but keep in mind there are many examples of an algebraic field. It is assumed in the axioms for a field that the additive identity 0 and the multiplicative identity 1 are not equal.

[3] We often drop the multiplication symbol "$\cdot$" and denote multiplication of two elements as $a\cdot b=ab$.

subtraction: $a-b=a+(-b)$ (read a minus b)

division: $\frac{a}{b}=ab^{-1}$ (for $b\neq 0$) (read a divided by b)

Important Note: A field is an algebraic system where you can add, subtract, multiply and divide (except by 0) in the same manner you did as a child. As a child, you were taught these were "properties" of numbers. But they are not their properties, they are the definition or rules of engagement of the real numbers. A subtle, but important point.

We know what you are thinking: you have known all this since 3rd grade. If your argument is that the axioms of arithmetic are simple and elementary, that is no argument at all. Axioms are supposed to be simple. The question you should ask is what kind of results can be proven from the axioms. The answer is there are many deep results for an algebraic field. Ask yourself if these are the simplest axioms that give rise to a system of arithmetic? Do you need any more axioms? Can you get by with less? These are not trivial questions and their answers are even less so. There are other systems of axioms that allow you to perform "arithmetic" operations on elements of a set, such as groups and rings that we will learn about in Chapter 6.

Fields other than $\mathbb{R}$

1. **Boolean Field:** Let $F_2=\{0,1\}$ and define the binary operations of addition and multiplication by the following table.

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

The set A with these arithmetic operations is an algebraic field. We leave it to the reader to check all the properties a field must possess.

2. **Complex Numbers:** The complex numbers $a+bi$, where a,b are real numbers and $i=\sqrt{-1}$, where addition and multiplication are defined in the usual manner.

2. **Rational Numbers $\mathbb{Q}$:** The rational numbers where addition and multiplication are defined in the usual way.

4. **Rational Functions F :** The set of all rational functions

$$f(x)=\frac{p(x)}{q(x)}$$

where $p(x), q(x) \neq 0$ are polynomials with real coefficients, where addition and multiplication are defined in the usual way and 0 and 1 are the standard additive and multiplicative identities.

There are many other examples of fields studied by mathematicians, including the Galois finite fields, *p*-adic number fields, and fields of functions, such as meromorphic and entire functions.

We now come to the second group of the three types of axioms required to describe the real numbers.

Ordered Fields

Although an algebraic field allows us to carry out arithmetic on a set, what it cannot do is compare the size of members of the set. The job now is to include "order" on the field. To do this, we split the field into two disjoint sets, *P* and *N*, called the **negative** and **positive** members of the field. These two sets mimic the properties of the positive and negative real numbers. This motivates the general definition of an ordered (algebraic) field.

Definition: Ordered Field: An algebraic field F is said to be ordered if its **nonzero members** can be split into two **disjoint subsets**, $F = P \cup N$ called respectively, the **negative** (N) and **positive** (P) members in such a way that

- $x, y \in P \Rightarrow x + y \in P$
- $x, y \in P \Rightarrow xy \in P$
- $x \in P \Leftrightarrow -x \in N$

These properties allow us to define a **strict (total) order** $<$ on F by

$$x < y \Leftrightarrow y - x \in P$$

which is a total order on F since

- irreflexive since $x - x = 0 \notin P \Rightarrow x \not< x$
- asymmetric $x < y \Rightarrow \sim (y < x)$
- transitive since $[(x < y) \wedge (y < z)] \Rightarrow x < z$

For convention, we say

- $x > 0$ when $-x < 0$
- $x \geq 0$ when $x > 0$ or $x = 0$
- $x \leq 0$ when $x < 0$ or $x = 0$

We can now prove all the usual properties of inequalities of real numbers.

Important Note: One does not just make rules or axioms willy-nilly hoping good things will follow. In fact, it's just the opposite. Knowing what is desirable, one designs axioms from which the desirable theorems will follow. As the mathematician Oswald Veblen once said, “The test of a good axiom system lies in the theorems it produces.”

Historical Note: The concept of an ordered field was introduced by Austrian/American mathematician Emil Artin (1898-1962) in 1927. Artin was one of the leading algebraists of the 20th century who emigrated to the U.S. in 1937 and spent many years at Indiana University and Princeton University.

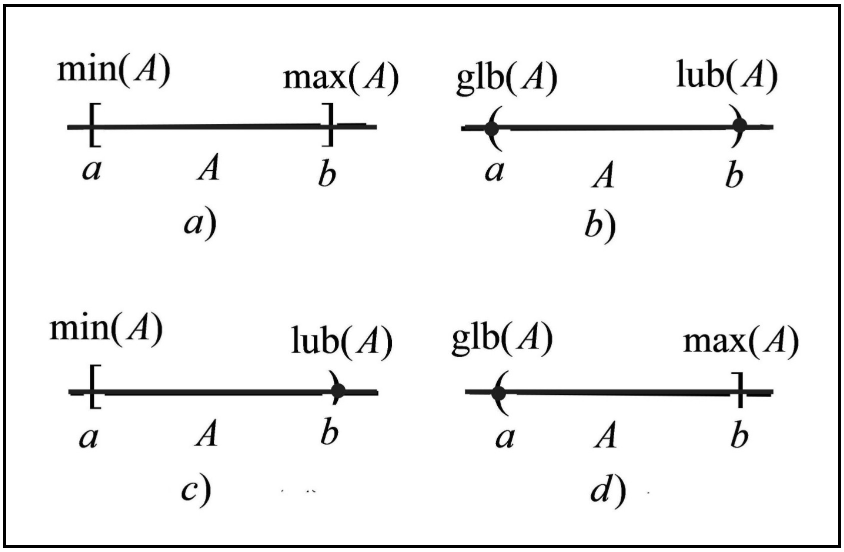
The Completeness Axiom

If we were to stop with ordered fields, we would be neglecting that special ingredient that defines the real numbers as a *continuum.* There are many examples of ordered fields that are not the real numbers, and all those algebraic systems have “gaps” between their elements. The set of rational numbers is an ordered field, which as we all know, has gaps (an uncountable number) between its members, two gaps being the solutions of $x^2 = 2$, which are $x = \pm\sqrt{2}$, which was proven long ago is not a rational number. What we need is an axiom that “fills in” these gaps and this is where the *completeness* (or *continuum*) axiom comes into play.

An interesting aspect of the completeness axiom is that over the years mathematicians have found several completeness axioms that are logically equivalent. Thus, it is possible to introduce any one of them as the "completeness" axiom. In this book, we have chosen the completeness axiom as the **least upper bound** axiom, because several interesting concepts can be easily deduced by working with it. Then, there is the other benefit, it is easy to understand. Before stating the axiom, however, we review a few important ideas about the least upper bound of a set introduced in Section 3.2.

Least Upper Bound and Greatest Lower Bounds

We use the four intervals in Figure 1 as a prop for reviewing the concepts of the least upper bound (lub) and the greatest lower bound (glb) introduced in our study of orders in Section 3.2.



Max, min, lub, glb
Figure 1

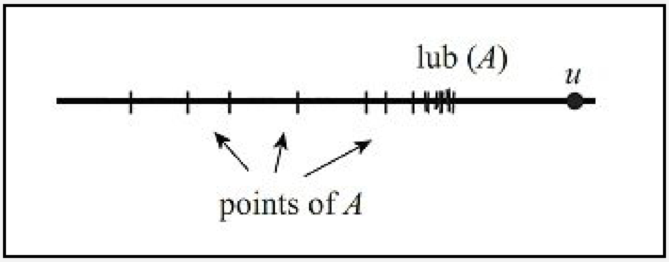
The intervals $(a,b),[a,b],[a,b)$, and $(a,b]$ are all bounded, both above and below. Bounded above simply means there is at least one number greater than or equal to all the elements in the set. Likewise, a lower bound for a set is a number less than or equal to all the elements in the set. Of course, not all sets are bounded; the set $[1,\infty)$ is bounded below but not above, and $(-\infty,\infty)$ is neither bounded above nor below. The intervals $[a,b]$ and $(a,b]$ each have a maximum value of b, whereas the intervals (a,b) and $[a,b)$ do not have a maximum value. The same arguments hold for minimum values. The intervals $[a,b]$, $[a,b)$ each has a minimum value of a, but the intervals (a,b) or $(a,b]$ do not have minimum values.

So what is the meaning $\text{lub}(A)$ and $\text{glb}(A)$ in Figure 1? Note that two of the intervals contain their maximum value and two do not. However, and this is the important part, for *each* of the four intervals $[a,b]$, $[a,b)$, (a,b), and $(a,b]$, the set of upper bounds is the *same,* namely $[b,\infty)$, and note that this set of upper bounds contains its minimum of b. For the intervals (a,b), $[a,b)$ where b does not belong to the interval, we call b the **least upper bound** (or **supremum)** of the set since it is the least of the upper bounds of the set. We denote this value by lub(A). For the two sets $[a,b]$ and $(a,b]$ that *have* a maximum value, the least upper bound of the set is the same as the maximum of the set. For the sets (a,b) and $[a,b)$ that do not have maximum values, the least upper bound b is a kind of "surrogate" for the maximum.

The same principle holds for lower bounds. The set of lower bounds for the four intervals is the same, namely $(-\infty,a]$. The number a is the greatest of all these lower bounds and is called the glb(A) for each of the four intervals. Any set that is bounded below may or may not have a *minimum* value, but the set of lower bounds will always have a *maximum* value, and that maximum value is called the **greatest lower bound** (or **infimum**) of the set and denoted by glb(A).

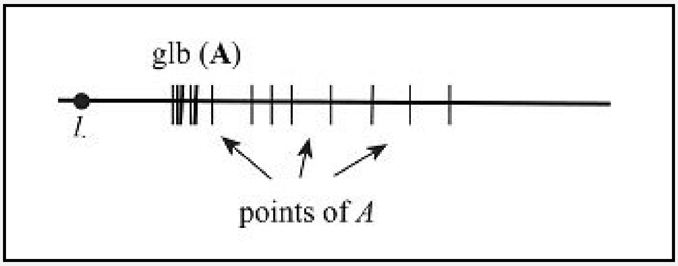
Defintion Let A be a set in an ordered field that is bounded above. The number $\text{lub}(A)$ is the **least upper bound** or **supremum** of A if

- $\text{lub}(A)$ is an upper bound of A, *i.e.* $x \le \text{lub}(A)$ for all $x \in A$..
- if u is any upper bound for A, then $\text{lub}(A) \le u$.



Likewise the number $\text{glb}(A)$ is the **greatest lower bound** or **infimum** of A if

- $\text{glb}(A)$ is a lower bound of A, *i.e.* $\text{glb}(A) \le x$ *for all* $x \in A$..
- if l is any lower bound for A, then $l \le \text{glb}(A)$.



This leads us to the completeness axiom for $\mathbb{R}$, which up to now, we have endowed with field and order axioms. The last set of axioms we assign to $\mathbb{R}$ (actually only one axiom) is called the completeness axiom, which we use the least upper bound version.

Completeness Axiom: Least Upper Bound Axiom[4]

[4] In Section 4.1 we defined the real numbers in terms of rational numbers by Dedekind cuts. Although we did not prove it at that time, the real numbers defined by Dedekind cuts satisfies the completeness axiom.

In an ordered field, if every non-empty set that is bounded above has a least upper bound, then the ordered field satisfies the **completeness axiom**.

We are now (finally) ready to define the real numbers.

Definition of the Real Numbers The **real number system** $\mathbb{R}$ is a **complete ordered field**, that is, an ordered field that satisfies the completeness axiom. Stated another way, it is a set $\mathbb{R}$ that satisfies the axioms of an **algebraic field**, the **order axioms**, and the **completeness axiom**.

The least upper bound axiom is necessary since there are ordered fields that do not "look like" the real numbers and the reason is that they don't satisfy the least upper bound axiom. Of the ordered fields that do not satisfy the completeness axiom, the rational numbers are the most well-known. By including the completeness axiom with an ordered field, the ordered field behaves like the real line you have been studying your entire mathematical career.

When we refer to the real numbers as a complete ordered field, we always say *the* complete ordered field all complete ordered fields are *isomorphic*. We say that two abstract structures are isomorphic if they have exactly the same mathematical structure and differ only in the symbols used to represent various objects and operations in the system.

Example 1: Rational Numbers and the Completeness Axiom Do the rational numbers satisfy the completeness axiom?

Solution: In order for the rational numbers to satisfy the completeness axiom, every set of rational numbers that is bounded above must have a least member. The rational numbers fail the completeness axiom since the set

$$A = \left\{ q \in \mathbb{Q} : 0 < q < \sqrt{2} \right\}$$

of rational numbers is bounded above (5 is an upper bound), but it has no least upper bound since the set of upper bounds

$$\text{set of upper bounds } = \left[\sqrt{2}, \infty \right)$$

has no smallest (rational) member.

Historical Note: Czech mathematician Bernard Bolzano (1781-1848) conceptualized the least upper bound property of the real numbers in an 1817 paper in which he gave the first *analytic* proof of the Intermediate

Value Theorem. He realized the proof must depend on deep properties of the real numbers.

Important Note: In the previous Section 4.1 when we defined the real numbers in terms of Dedekind cuts of rational numbers, we did not show how an ordering could be defined on the real numbers in terms of Dedekind cuts, or did we show that the real numbers were complete in terms of Dedekind cuts. This could have been done, but it would have been a long and tedius task and so it was omitted. . .

Problems

Normally when we solve problems in elementary algebra, the rules of the game are so entrenched in our minds that we use them without much thought. Solve Problems 1-4 using the axioms of an algebraic field and tell which axioms is being used.

1. **Parentheses Not Needed in Addition** Show that in the sum of four real numbers $a,b,c,d \in \mathbb{R}$, parentheses are not required.

2. **Parentheses Not Needed in Multiplication** Show that for the product of four real numbers $a,b,c,d \in \mathbb{R}$, parentheses are not required.

3. **Solving a Middle School Equation** Show that $(\forall a,b \in \mathbb{R})$ the equation $a+x=b$ has exactly one solution, which is $x=b+(-a)$.

4. **A Negative of a Negative** Show that $(\forall a \in \mathbb{R})[a=-(-a)]$

5. **True or False**
   a) The natural numbers $\mathbb{N}$ with operations of addition and multiplication is an ordered field.
   b) $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields but $\mathbb{C}$ is not.
   c) For A,B bounded sets of real numbers, the identity

$$\sup(A-B)=\sup(A)-\sup(B)$$

   holds, where $A-B=\{a-b : a \in A,\ b \in B\}$.

   d) The integers $\mathbb{Z}$ constitute an ordered field.
   e) All finite sets of real numbers have a least upper bound.
   f) The rational numbers less than 1 have a least upper bound.
   g) If a subset of the real numbers has an upper bound, then it has exactly one least upper bound.

h) $\sup(\mathbb{Z}) = \infty$
i) Every finite set can be ordered.
j) The set of linear functions $f(x) = ax + b$ with the usual addition and multiplication of functions is an algebraic field.

k) In plain English, the completeness axiom ensures there are no "holes" in the real numbers.

6. **Glb, Lub, Max, and Min** If they exist, find max(A), min(A), lub(A), and glb(A) for the following sets.

a) $A = \{1,3,9,4,0\}$
b) $A = [0,\infty)$
c) $A = \{x \in \mathbb{Q} : 0 \le x < 1\}$
d) $A = [-1,3]$
e) $A = \{x : x^2 - 1 = 0\}$
f) $A = \{n \in \mathbb{N} : n \text{ divides } 100\}$
g) $A = \{x \in \mathbb{R} : x^2 < 2\}$
h) $A = (-\infty,\infty)$
i) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

7. **More Difficult Sup and Inf** If they exist, find the least upper bound and greatest lower bound of the set

$$A = \left\{\frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N}\right\}.$$

8. **Algebraic Field** Show that the rational numbers with the operations of addition and multiplication form an algebraic field.

9. **Boolean Field** Show that the set $F_2 = \{0,1\}$ consisting of 2 elements and following arithmetic operations forms an algebraic field.

+	**0**	**1**
0	**0**	**1**
1	**1**	**0**

×	**0**	**1**
0	**0**	**0**
1	**0**	**1**

10. **Ordered Field** Show that the rational numbers with the operations of addition and multiplication and usual ordering relation "less than" $"<"$ form an ordered field.

11. **Not an Ordered Field** Show that the complex numbers is an algebraic field but not an ordered field.

12. **Well-Ordering Principle** The **well-ordering principle**[5] states that every (nonempty) subset $A \subseteq \mathbb{N}$ contains a smallest element under the usual ordering $\leq$. Does this principle hold for all subsets $A \subseteq \mathbb{Z}$?

13. **Well-Ordering Theorem** A partial order $"\preceq"$ on a set X is called a **well-ordering** (and the set X is called **well-ordered**) if every nonempty subset $S \subseteq X$ has a least element. The **Well- Ordering Theorem**[6] states that every set can be well-ordered by some partial order. Are the following sets well ordered by the usual "less than or equal to" order $"\leq"$?

a) $\mathbb{N}$ Ans: yes
b) $\{3,4,5\}$ Ans: yes
c) $\mathbb{Z}$
d) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

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[5] This principle is a theorem and equivalent to the Principle of Mathematical Induction.
[6] The Well-Ordering Theorem is equivalent to the Axiom of Choice and was proven by the German mathematician Ernst Zermelo (1871-1953). Although the theorem says the real numbers $\mathbb{R}$ are well-ordered, no one has ever found a well-ordering.